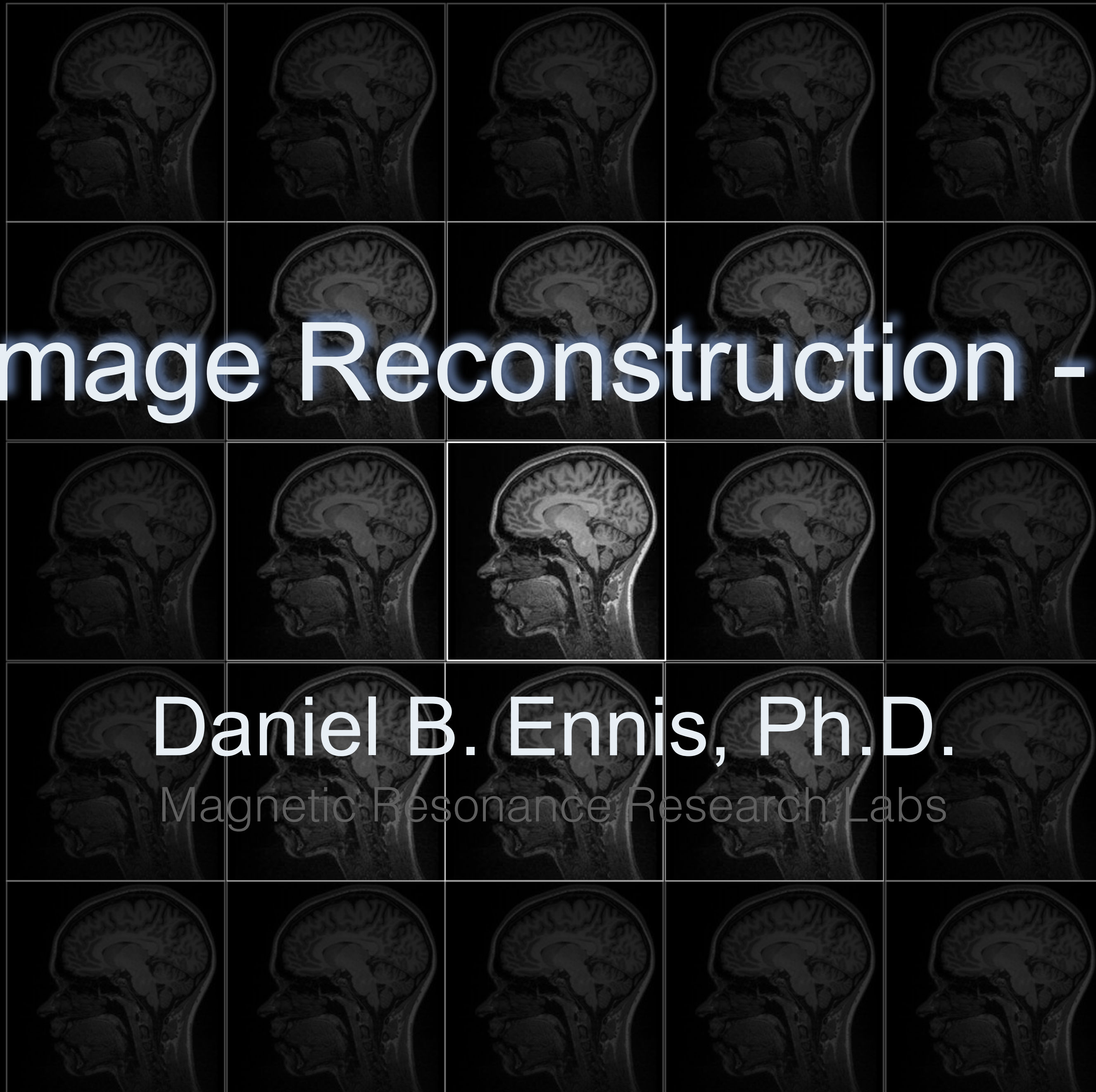
Image Reconstruction - I



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Lecture #11 - Learning Objectives

- Understand the small tip angle approximation.
- Appreciate that the small tip angle approximation works for intermediate flip angles!
- Understand what truncation artifacts are and one way to reduce them.
- Learn to describe *k*-space in words and mathematically.
- Appreciate what different points in *k*-space represent.
- Understand the connection between Fourier encoding and image acquisition.
- Be able to describe the roll of phase and frequency encoding.

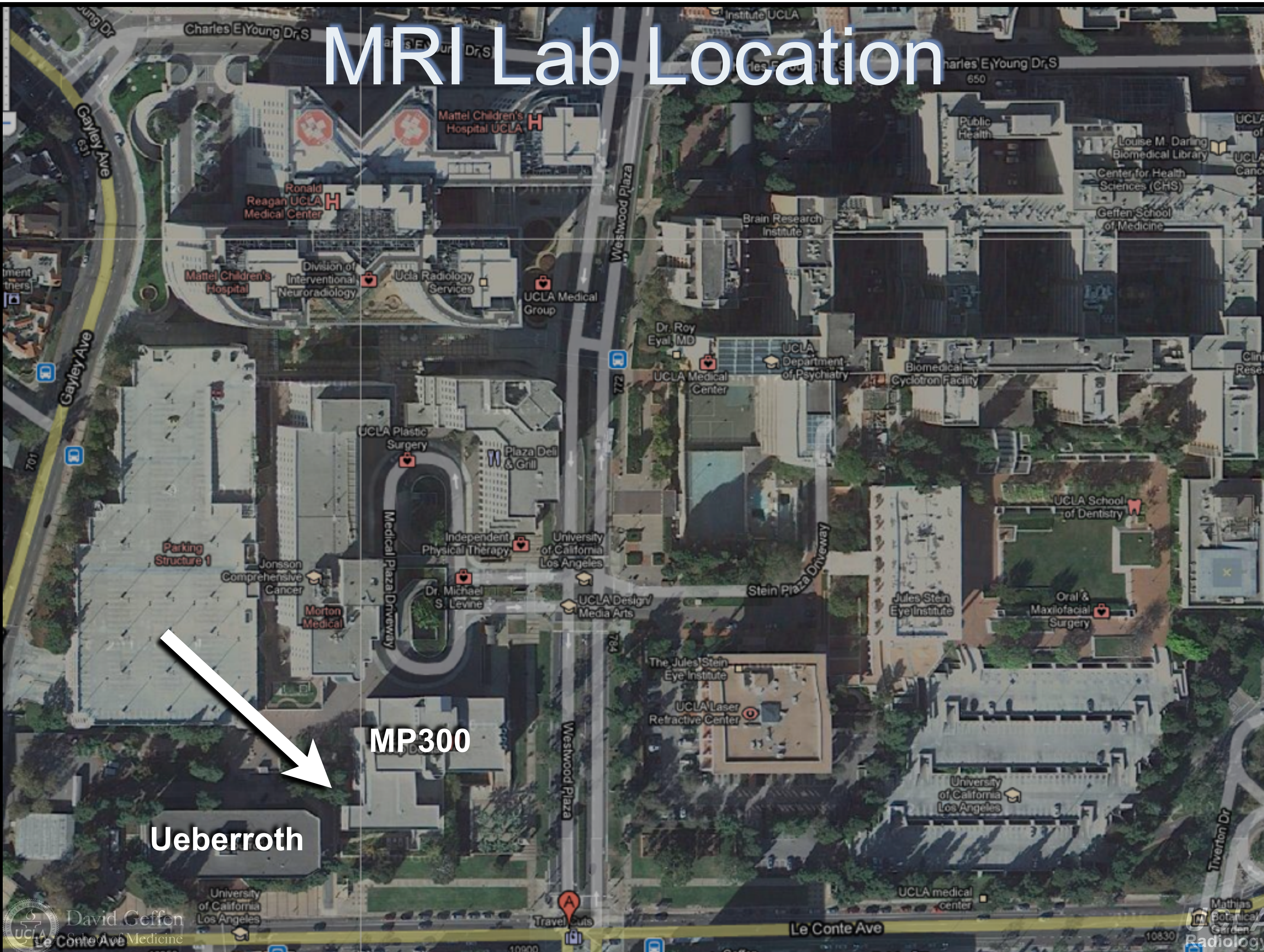
Class Business

- Thursday (2/23) from 6-9pm
  - 6:00-7:30pm Groups
    - Avanto
      - Binru Chen, Junjie Chen, Yuhua Chen
    - Skyra
      - Jie, Qihui, Cass
    - Prisma
      - Nyasha, Fadil, Vahid
  - 7:30-9:00pm Groups
    - Avanto
      - Sara, Yara, April
    - Skyra
      - Timothy, Diana, Zhaohuan, Xingmin (?)
    - Prisma
      - Daisong, Jingwen, Fang-Chu, Timothy
- BRING THE COMPLETED SCREENING FORM

Class Business

- **HW #1**
  - 13.3±3.2 [15.75,6.5]
- **HW #2**
  - 11.7±2.6 [15, 6]
- **Class Average**
  - 25.5±5.5 [30.5, 12.5]
- **<20 points please see me…**

MRI Lab Location



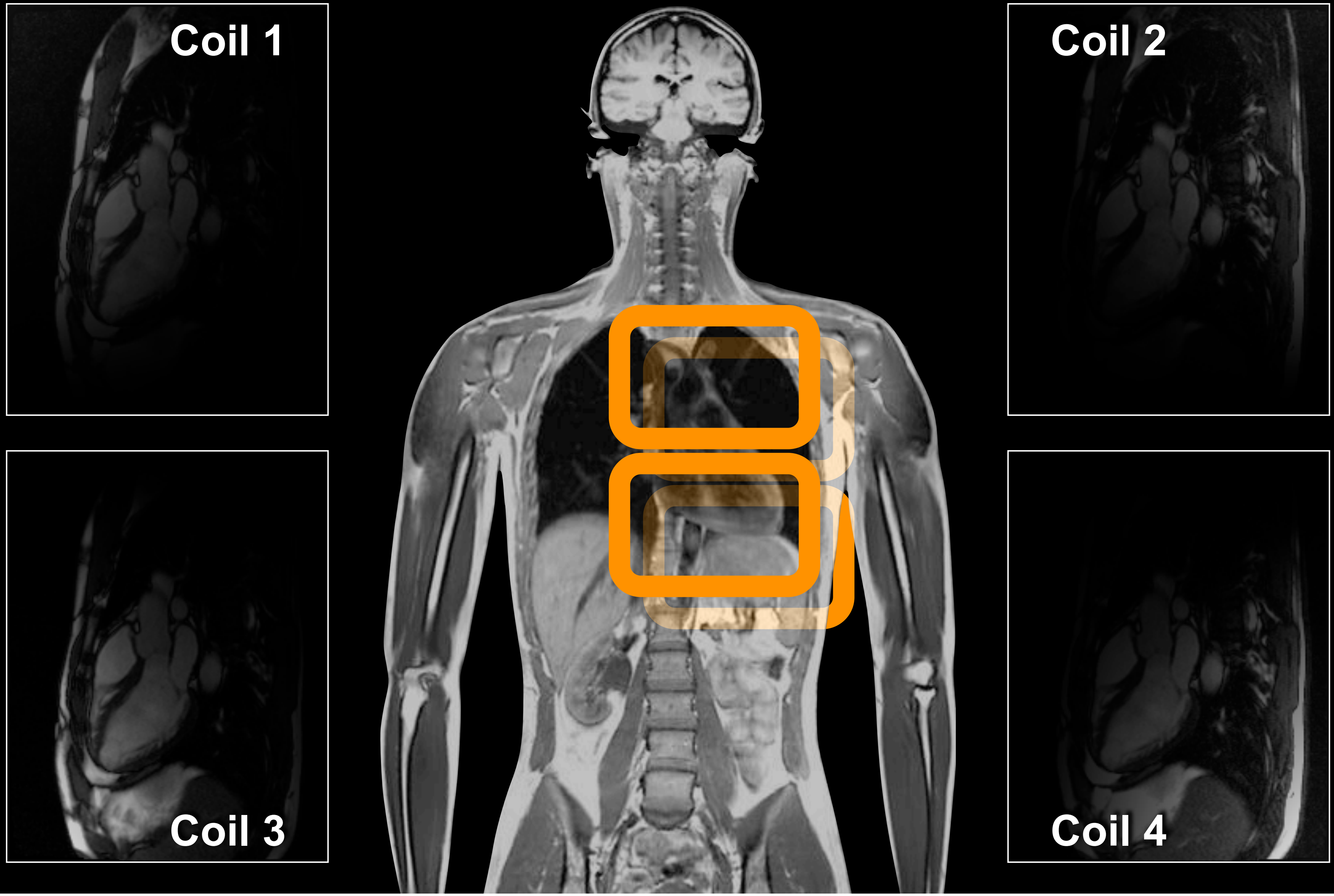
Lecture #13 - Learning Objectives

- **Understand how to combine data from several receiver channels.**
- **Appreciate how the final image is obtained from the sum over all sampled spatial frequency (Fourier) patterns.**
- **Define how the field-of-view and the number of acquired data points impacts spatial resolution.**
- **Describe the parameters that control the field of view.**
- **Understand the applications of zero padding and windowed reconstructions.**
- **Identify sources of Gibb's ringing.**

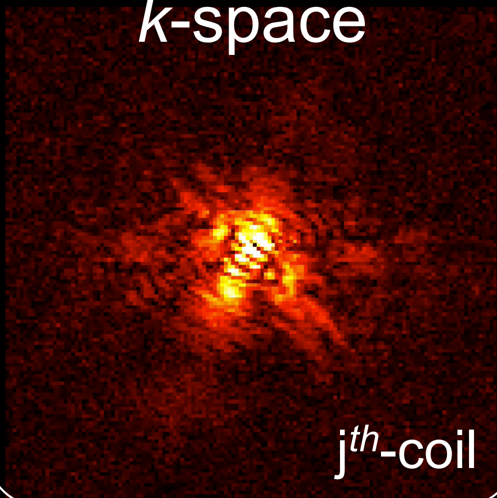
Multi-Channel Reconstruction

Multiple Coil Reconstruction

Each coil element (channel) has a unique sensitivity profile – $\vec{B}_r\,(\vec{r})$



Multiple Coil Reconstruction

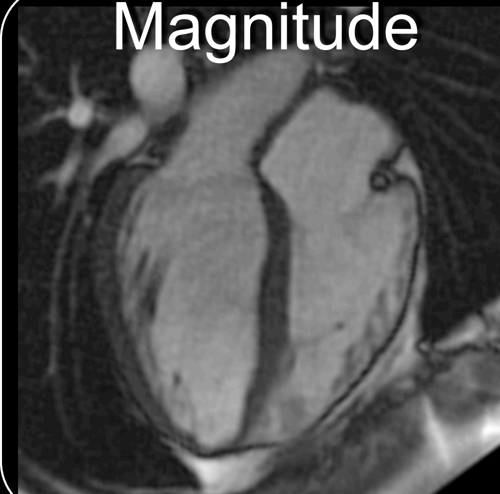


FFT^{-1} →

RMS

$$I(\vec{r}) = \sqrt{\sum_j \left(\frac{|I_j(\vec{r})|^2}{\sigma_j^2} \right)}$$

→



$I(\vec{r}) \longrightarrow$ Final ***magnitude*** image

$I_j\,(\vec{r}) \longrightarrow$ Image from jth coil

$\sigma_j^2 \longrightarrow$ Noise variance

- Depends on coil loading
- Proximity to patient
- Measured with "noise scan"
- Weights each coil's contribution

Image Reconstruction

Image Reconstruction

$$S = \mathcal{T}\{I\}$$

Data Consistency Constraint

Measured Signal

Spatial Information Encoding Scheme (Fourier Transform)

Image Function

$$I = \mathcal{T}^{-1}\{S\}$$

Our task is to recover *I* from the measured signals.

MR Signal Equation

$$s\left(t\right)=\int\int_{x,y}\vec{M}^{0}_{xy}\left(\vec{r}\right)\cdot e^{-i\Delta\omega\left(\vec{r}\right)t}\mathrm{d}\vec{r}$$

The MRI Signal Equation is the…

$$s\left(t\right)=\int\int_{x,y}\vec{M}^{0}_{xy}\left(x,y\right)\cdot e^{-i\Delta\omega\left(x,y\right)t}\mathrm{d}x\mathrm{d}y$$

…2D Fourier Transform!

$$\Delta\omega(x,y)=\gamma G_x\cdot x+\gamma G_y\cdot y$$

Gradients define Δw

$$k_x(t)=\frac{\gamma}{2\pi}G_x t \qquad k_y(t)=\frac{\gamma}{2\pi}G_y t$$

k-space is convenient…

$$s\left(k_x(t),k_y(t)\right)=\int\int_{x,y}\underbrace{\vec{M}^{0}_{xy}\left(x,y\right)}_{I\left(\vec{r}\right)}\cdot e^{-i2\pi\left[k_x(t)x+k_y(t)y\right]}\mathrm{d}x\mathrm{d}y$$

The Fourier Transform

$$S(\vec{k}) = \int_{-\infty}^{+\infty} I\left(\vec{r}\right) e^{-i2\pi \vec{k}\cdot\vec{r}} d\vec{r}$$

MRI Signal Equation

$$S(\vec{k}) \stackrel{\mathcal{F}}{\longleftrightarrow} I(\vec{r})$$

$$S(k_x) = \int_{-\infty}^{+\infty} I\left(x\right) e^{-i2\pi(k_x x)} dx$$

1D
Eqn. 5.93

$$S(k_x, k_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I\left(x, y\right) e^{-i2\pi(k_x x + k_y y)} dx dy$$

2D
Eqn. 5.98

$$S(k_x, k_y, k_z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I\left(x, y, z\right) e^{-i2\pi(k_x x + k_y y + k_z z)} dx dy dz$$

3D
Eqn. 5.110

Image Reconstruction

Given $S(\vec{k}_n) = \int_{-\infty}^{+\infty} I\left(\vec{r}\right) e^{-i2\pi \vec{k}_n \cdot \vec{r}} d\vec{r}$ MRI Signal Equation

How do we determine $I\left(\vec{r}\right)$?

Image Reconstruction

$$S(\vec{k}_n) = \int_{-\infty}^{+\infty} I\left(\vec{r}\right) e^{-i2\pi \vec{k}_n \cdot \vec{r}} d\vec{r}$$

MRI Signal Equation

$$\mathcal{D} = \left\{ \vec{k}_n = n\Delta\vec{k}, n = \ldots, -2, -1, 0, 1, 2, \ldots \right\}$$

Uniform *k*-space sampling

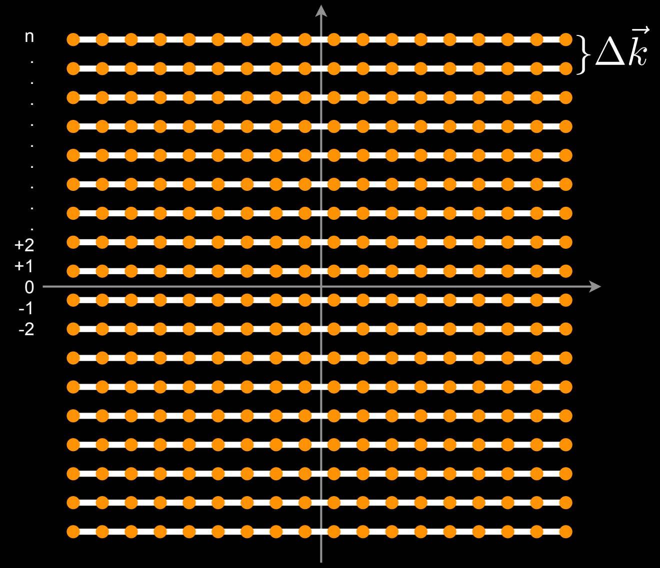


Image Reconstruction

$$S(\vec{k}_n) = \int_{-\infty}^{+\infty} I\left(\vec{r}\right) e^{-i2\pi \vec{k}_n \cdot \vec{r}} d\vec{r}$$

$$\mathcal{D} = \left\{ \vec{k}_n = n\Delta \vec{k}, n = ..., -2, -1, 0, 1, 2, ... \right\}$$

Uniform *k*-space sampling

$$S\left[n\right] = S\left(n\Delta k_x\right) = \int_{-\infty}^{+\infty} I\left(x\right) e^{-i2\pi n\Delta k_x \cdot x} dx$$

One-dimensional Case

Image Reconstruction

$$S\left[n\right] = \underbrace{S\left(n\Delta k_x\right)}_{\text{This is what we measure!}} = \int_{-\infty}^{+\infty} \underbrace{I\left(x\right)}_{\text{This is what we want!}} e^{-i2\pi n\Delta k_x \cdot x} dx$$

Image Reconstruction

$$\underbrace{S[n] = S(n\Delta k_x)}_{} = \int_{-\infty}^{+\infty} \underbrace{I(x)}_{} e^{-i2\pi n\Delta k_x \cdot x}\, dx$$

Eqn. 6.9

This is what we measure!

This is what we want!

We can show the following...(Page 191 in Lauterbur).

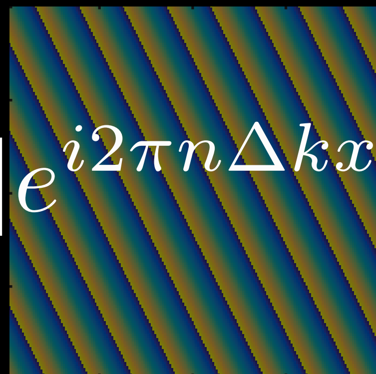
$$\sum_{n=-\infty}^{\infty} S[n] e^{i2\pi n\Delta k x} = \frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} I\left(x - \frac{n}{\Delta k}\right)$$

Eqn. 6.10

Fourier Series

Periodic Extension of I(x)

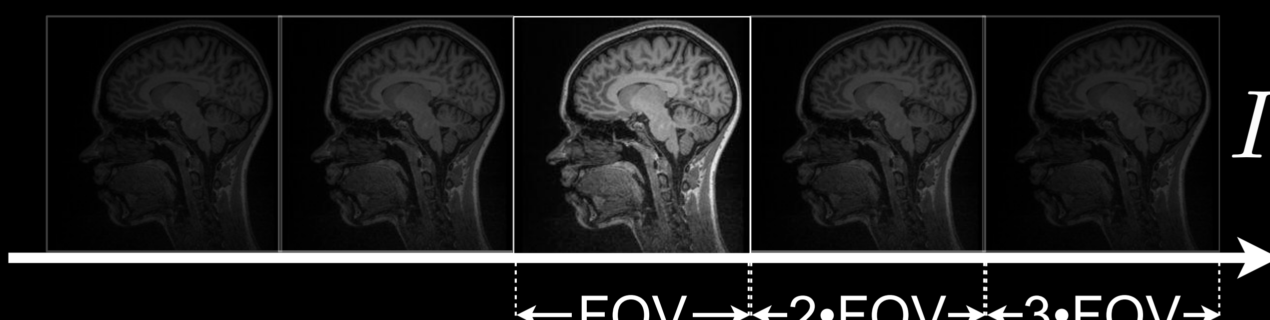
Image Reconstruction



$$\sum_{n=-\infty}^{\infty} S[n] e^{i2\pi n \Delta k x} = \frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} I\left(x - \frac{n}{\Delta k}\right)$$

- Fourier series
- Δk is the fundamental frequency
- S[n] coefficient of the nth harmonic

- Periodic extension of $I(x)$
- n is an integer
- Period is 1/Δk=FOV



$$I\left(x - \frac{n}{\Delta k}\right)$$

Periodic extensions of a object/function.

Infinite Sampling

$S(k)$ is measured at $k \in \mathcal{D}$

$$\mathcal{D} = \{n\Delta k, -\infty < n < +\infty\}$$

Infinite Sampling

$S(k)$ is measured at $k \in \mathcal{D}$

$$\mathcal{D} = \{n\Delta k, -\infty < n < +\infty\}$$

Can *I(x)* be recovered from its periodic extension?

$$\sum_{n=-\infty}^{\infty} S[n] e^{i2\pi n \Delta k x} = \frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} I\left(x - \frac{n}{\Delta k}\right)$$

Infinite Sampling

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$$\sum_{n=-\infty}^{\infty} S[n] e^{i2\pi n\Delta k x} = \frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} I\left(x - \frac{n}{\Delta k}\right)$$

$$\text{If } I(x) = 0 \text{ on } |x| > FOV_x/2 \left(i.e.\ \Delta k < \frac{1}{FOV_x}\right), \text{ then}$$

Infinite Sampling

$S(k)$ is measured at $k \in \mathcal{D}$

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Can *I(x)* be recovered from its periodic extension?

$$\sum_{n=-\infty}^{\infty} S[n]e^{i2\pi n\Delta kx} = \frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} I\left(x - \frac{n}{\Delta k}\right) \quad \text{Eqn. 6.10}$$

$$\text{If } I(x) = 0 \text{ on } |x| > FOV_x/2 \left(i.e.\ \Delta k < \frac{1}{FOV_x}\right), \text{ then}$$

$$I(x) = \Delta k \sum_{n=-\infty}^{\infty} S[n]e^{i2\pi n\Delta kx}, \ |x| < \frac{1}{\Delta k} \quad \text{Eqn. 6.16}$$

But ∞ takes forever...

Finite Sampling

$S(k)$ is measured at $k \in \mathcal{D}$

$$\mathcal{D} = \{n\Delta k, -N/2 \le n \le +N/2\}$$

Fourier Step-size

Number of Sample Points

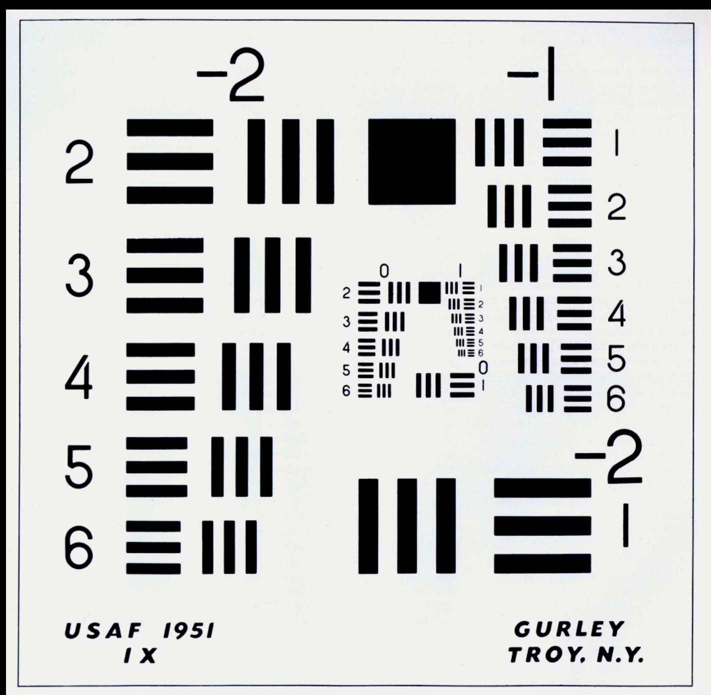
$$I(x) = \Delta k \sum_{n=-N/2}^{N/2-1} S[n] e^{i2\pi n \Delta k x}, \; |x| < \frac{1}{\Delta k}$$ Eqn. 6.20

This is the fundamental image reconstruction equation for MRI.

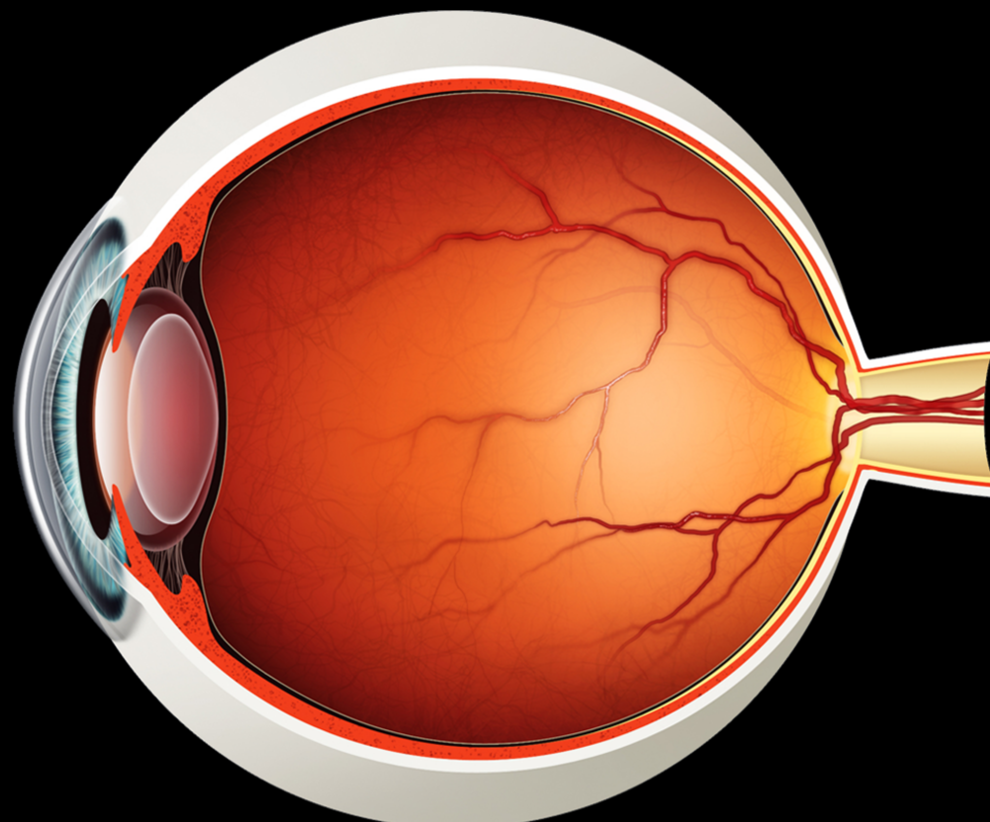
Spatial Resolution

Human Vision System

- What resolution can we see at?
  - 4-5 cycles per millimeter unaided
- How many “pixels” fill our visual field?
  - Order of 10e6 to 100e6



USAF Resolution Target



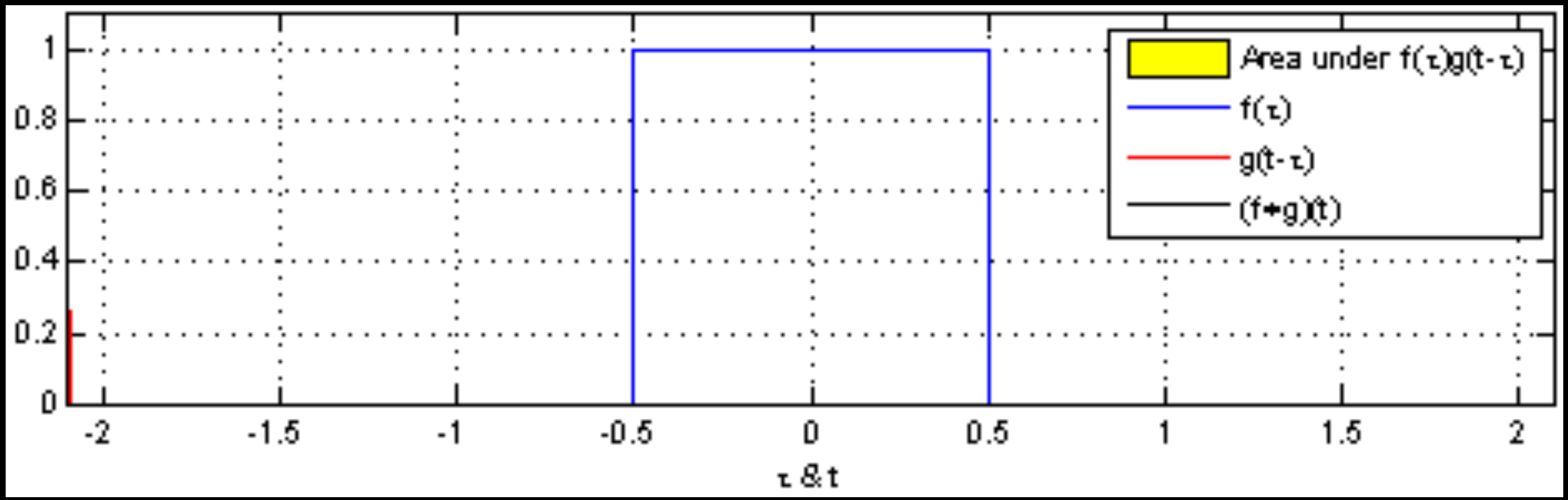
Spatial Resolution

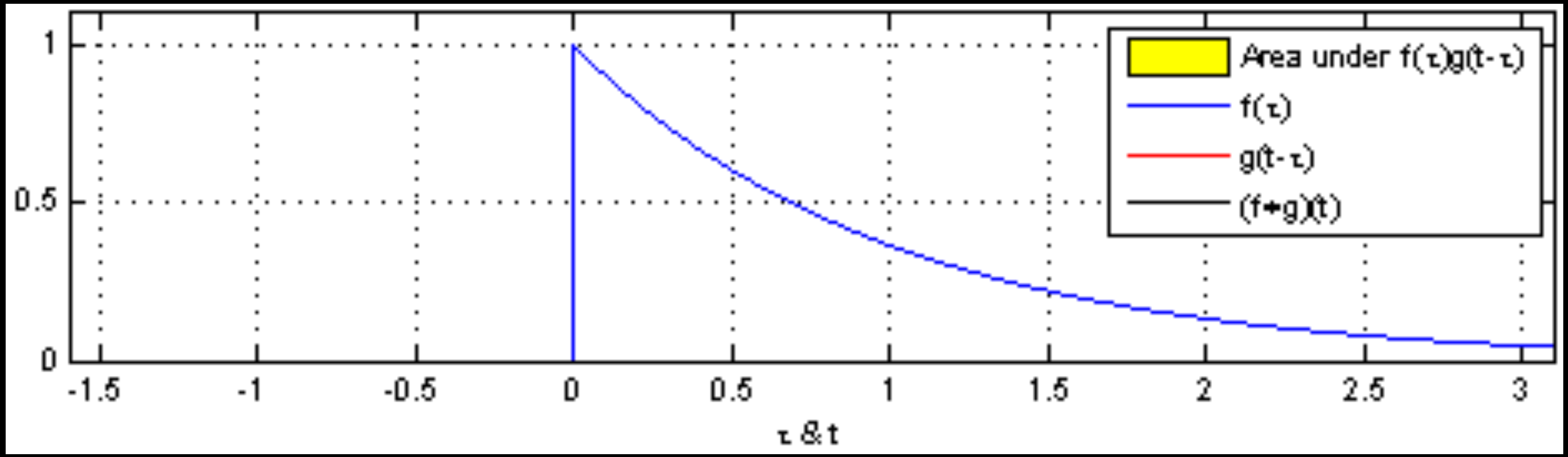
- **Spatial resolution of an imaging system is the smallest separation δx of two point sources necessary for them to remain resolvable in the resultant image.**

$$\hat{I}\left(x\right) = I\left(x\right) * h\left(x\right)$$

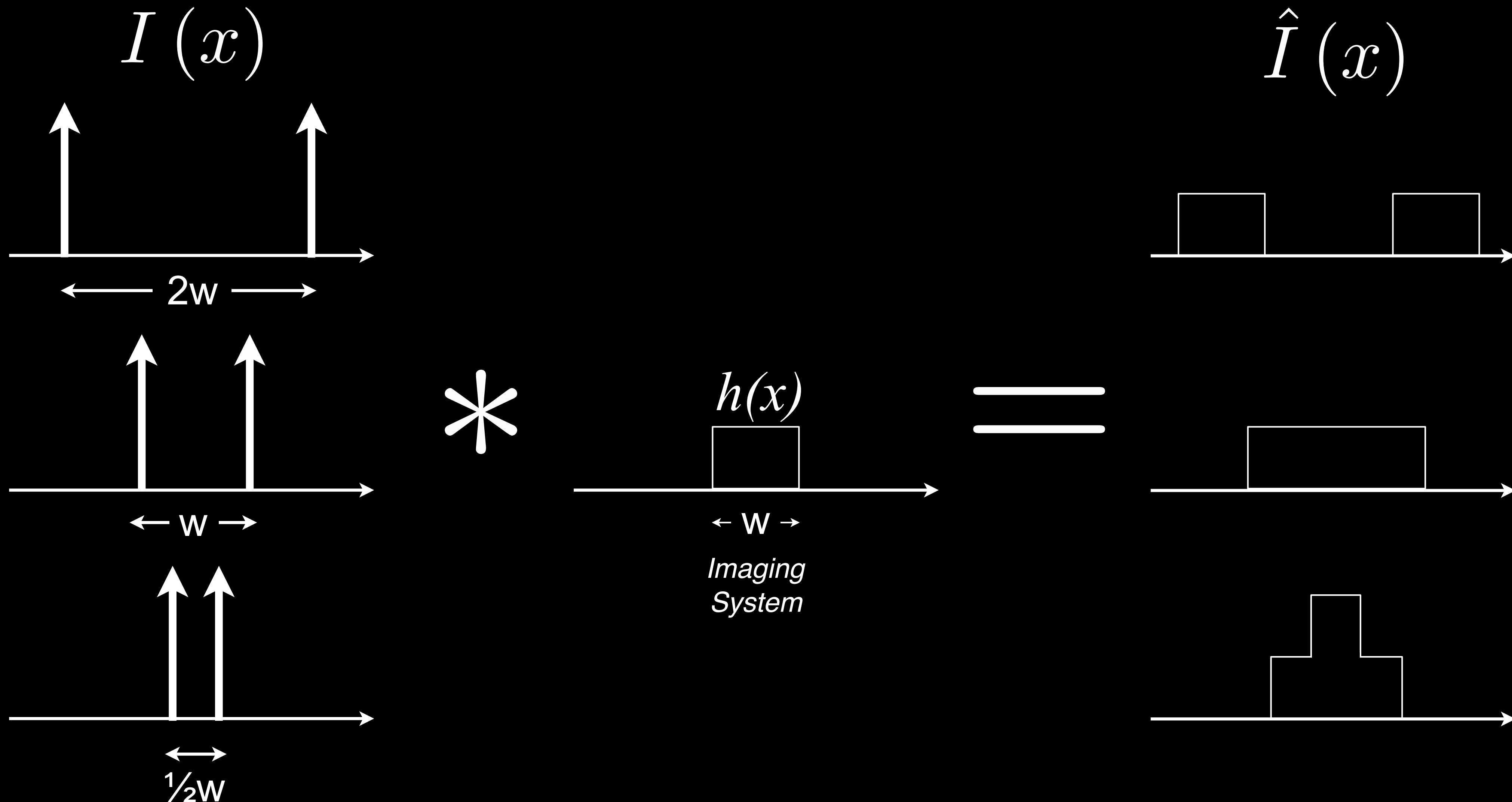
Image — Object — Point Spread Function

Convolution





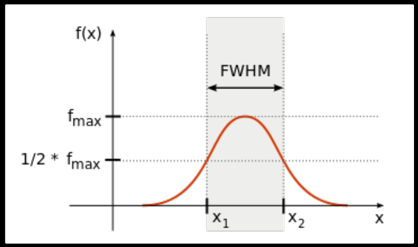
Spatial Resolution



$$\hat{I}(x) = I(x)\text{, if and only if } h(x) = \delta(x)$$

Spatial Resolution

- **The resolution limit of an imaging system is the width (W_h) of its point spread function:**
  - W_h **is the full-width half-max of** $h(x)$



Spatial Resolution

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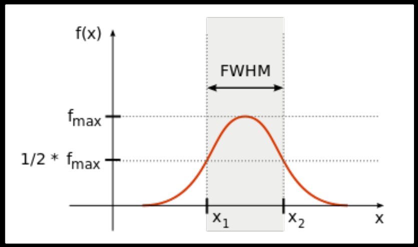


- **Alternately,**
  - W_h **of** $h(x)$ **is the width of an approximating box-function with the same height and area as** $h(x)$**:**

$$W_h = \frac{1}{h_{max}} \int_{-\infty}^{+\infty} h(x)dx$$

Point Spread Function

- **How do we determine the PSF, $h(x)$?** $\hat{I}(x) = I(x) * h(x)$
  - **Set $I(x)$ to be a δ-function, then**

$$\hat{I}(x) = h(x)$$

  - **Recall,**

$$\hat{I}(x) = \Delta k \sum_{n=-N/2}^{N/2-1} S[n] e^{i2\pi n \Delta k x}$$

Eqn. 6.20 / Eqn. 8.5

  - **Therefore,**

$$h(x) = \Delta k \sum_{n=-N/2}^{N/2-1} e^{i2\pi n \Delta k x}$$

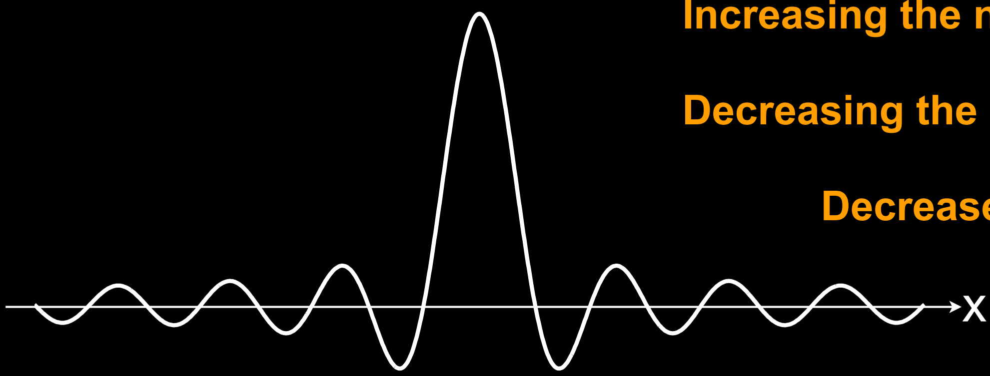
Eqn. 8.6

This is the PSF for Fourier sampling.

Fourier Reconstruction PSF

$$h(x) \approx \Delta k \frac{\sin(\pi N \Delta k x)}{\sin(\pi \Delta k x)} = Dir\left(N, \chi\right) \quad \text{Eqn. 8.7}$$

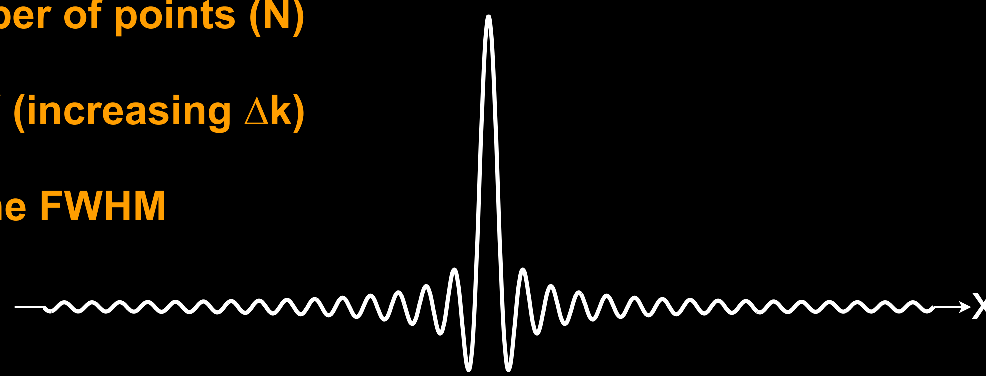
Dirichlet Function



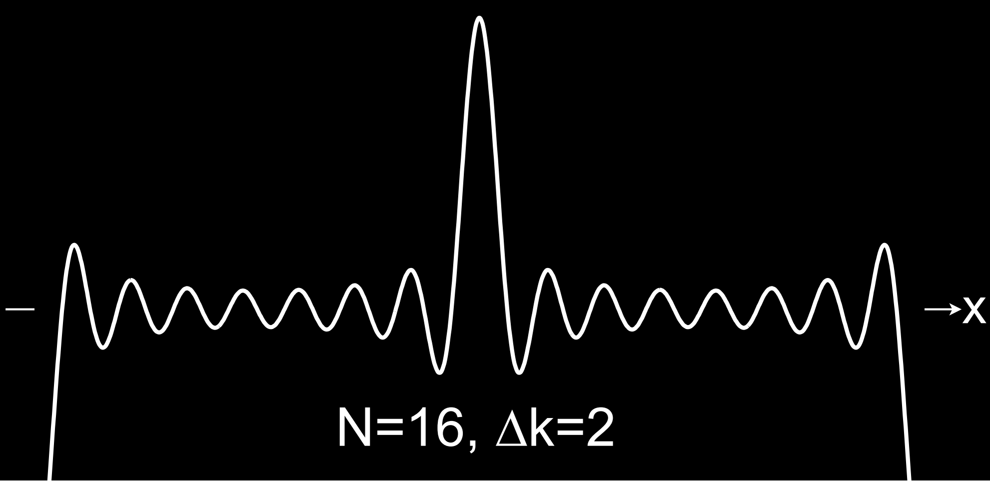
N=16, Δk=1

Increasing the number of points (N)
-or-
Decreasing the FOV (increasing Δk)

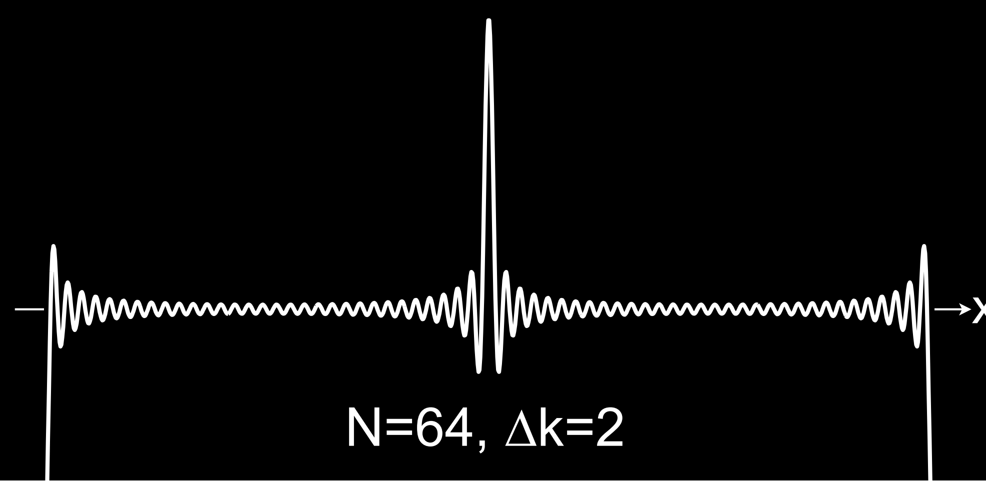
Decreases the FWHM



N=64,Δk=1



N=16, Δk=2



N=64, Δk=2

Fourier Reconstruction PSF

$$W_h = \frac{1}{h_{max}} \int_{-\frac{1}{2\Delta k}}^{\frac{1}{2\Delta k}} h(x)dx = \frac{1}{N\Delta k}$$ Eqn. 8.8

Limits over a single period

Fourier Pixel Size (Δx_F)

Fourier Reconstruction PSF

$$W_h = \frac{1}{h_{max}} \int_{-\frac{1}{2\Delta k}}^{\frac{1}{2\Delta k}} h(x)dx = \frac{1}{N\Delta k}$$

Limits over a single period

Fourier Pixel Size (Δx_F)

$$W_h = \frac{1}{N\Delta k} = \frac{FOV}{N}$$

Fourier Reconstruction PSF

$$W_h = \frac{1}{h_{max}} \int_{-\frac{1}{2\Delta k}}^{\frac{1}{2\Delta k}} h(x)dx = \frac{1}{N\Delta k}$$

Limits over a single period

Fourier Pixel Size (Δx_F)

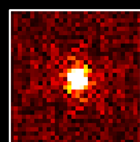
$$W_h = \frac{1}{N\Delta k} = \frac{FOV}{N}$$

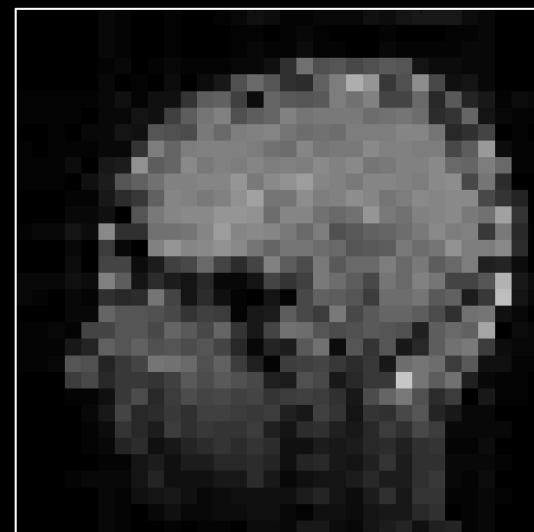
Note, we can't reduce W_h and N simultaneously, therefore

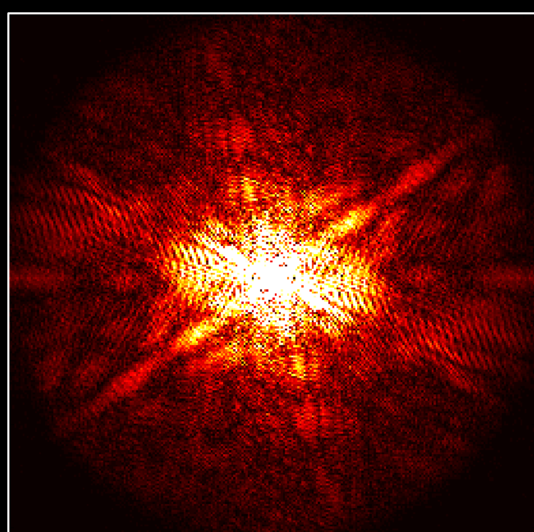
- An increase in spatial resolution (decrease in W_h) requires an increase in N or Δk (decrease in FOV)
- A decrease in spatial resolution (increase in W_h) requires a decrease in N or Δk (increase in FOV)

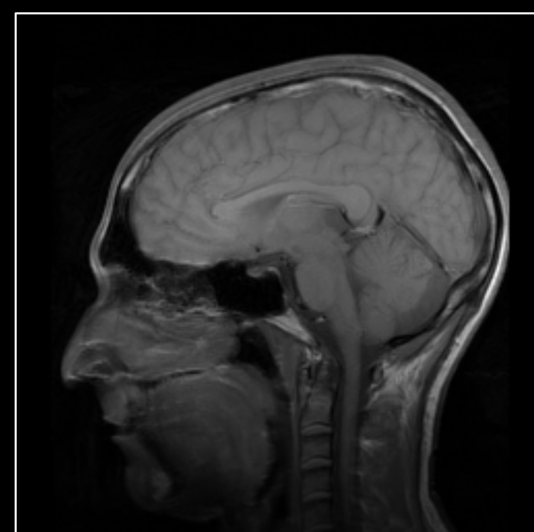
Finite Sampling

$$W_h = \frac{1}{N\Delta k} = \frac{FOV}{N}$$









Field of View

Sampling Theorem

- A space signal g(x) is <u>space-limited</u> if:
  - g(x)=0 for |x|>FOV/2
- A space signal g(x) is <u>band-limited</u> if:
  - its frequency spectrum is zero for $|k|>k_{max}$

Sampling Theorem

- A space signal g(x) is space-limited if:
  - g(x)=0 for |x|>FOV/2
- A space signal g(x) is band-limited if:
  - its frequency spectrum is zero for $|k|>k_{max}$
- If g(x) is:
  - Space-limited to |x|<FOV/2
  - Band-limited to $|k|<k_{max}$

Sampling Theorem

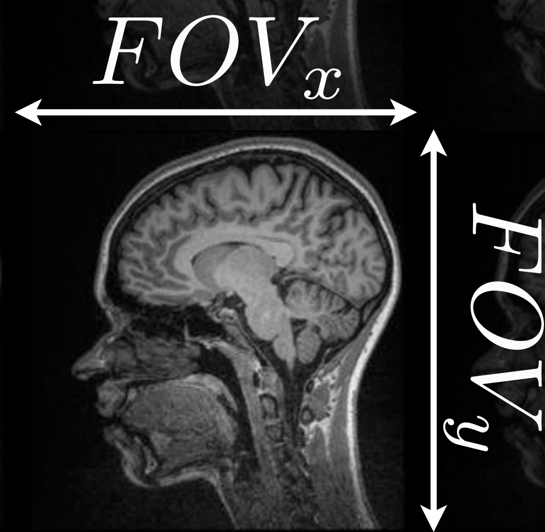
- A space signal g(x) is <u>space-limited</u> if:
  - g(x)=0 for |x|>FOV/2
- A space signal g(x) is <u>band-limited</u> if:
  - its frequency spectrum is zero for $|k|>k_{max}$
- If g(x) is:
  - Space-limited to |x|<FOV/2
  - Band-limited to $|k|<k_{max}$
- Then,

$$\Delta x = \frac{1}{N\Delta k} \quad \text{pixel size for} \quad k_{max} = N\Delta k$$

$$FOV_x = N\Delta x$$

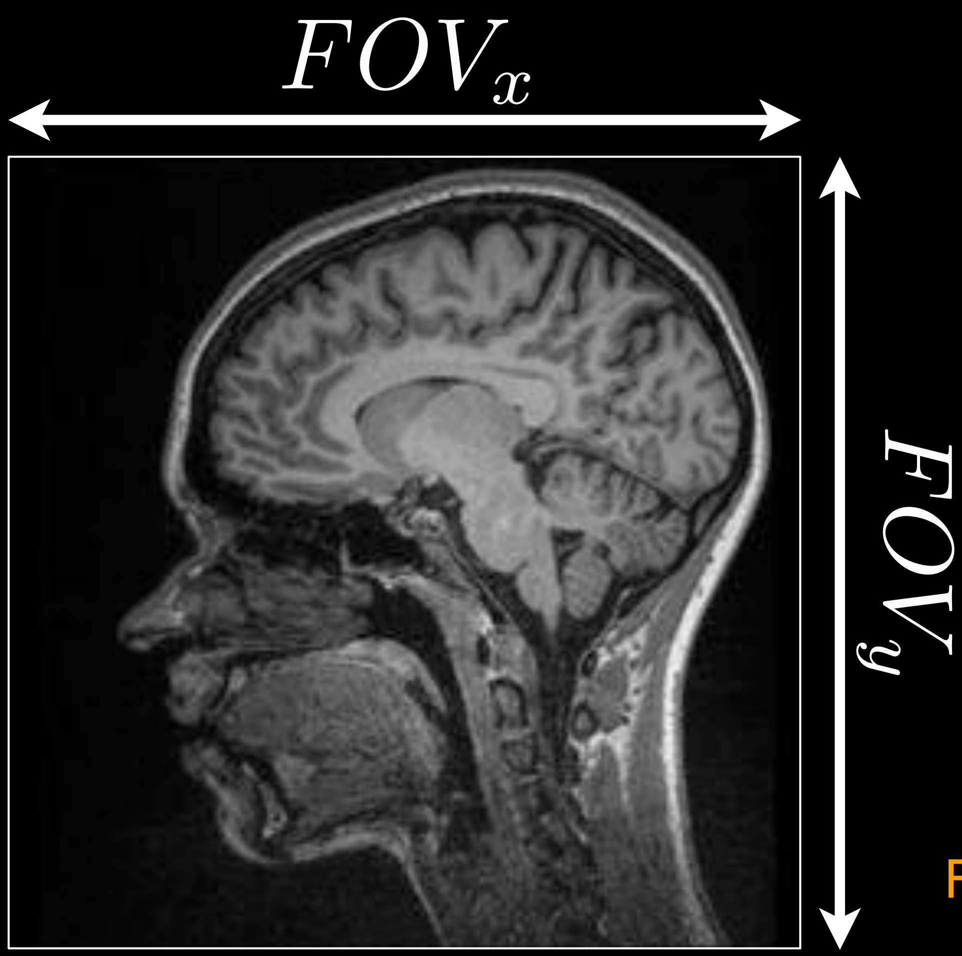
$$FOV_x = \frac{1}{\Delta k_x}$$

Field of View



- **The object repeats because...**
- **The Fourier summation series repeats, but**
- **We know the signal is space-limited,**
- **Therefore we truncate it.**

Field of View



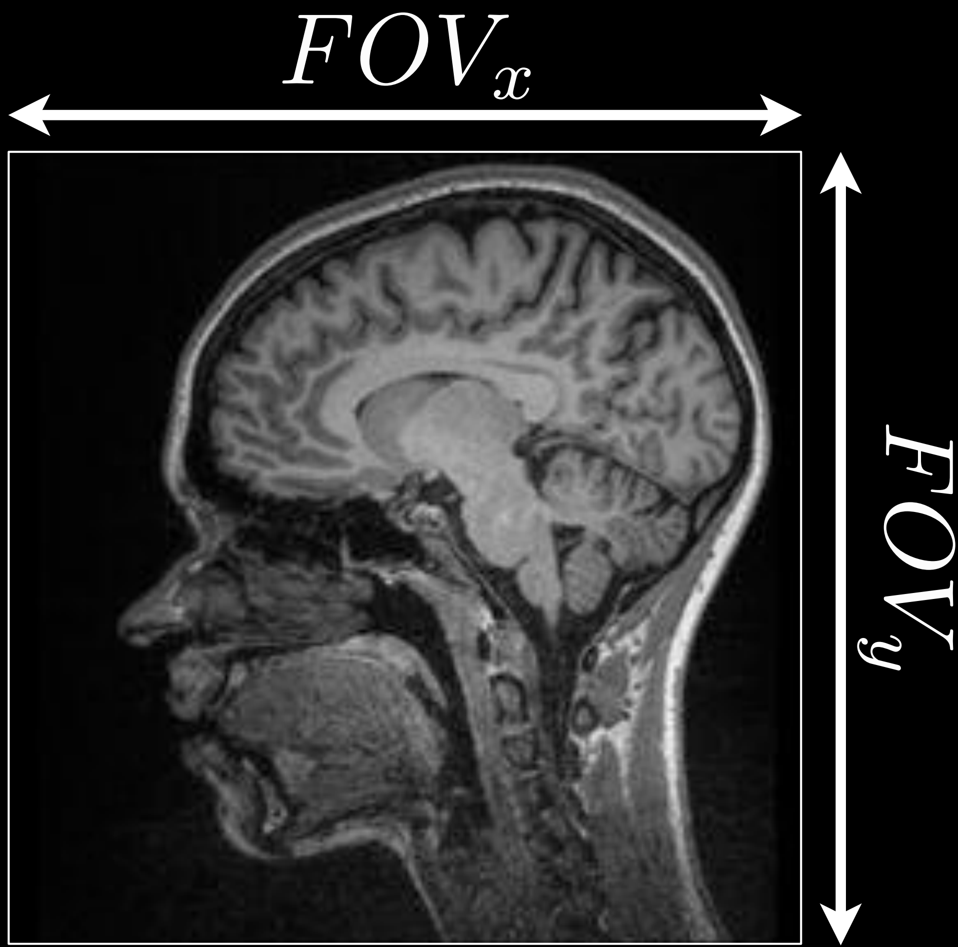
$$\Delta k_x = \frac{1}{\mathrm{FOV}_x} = \gamma |\mathbf{G}_x| \Delta t$$ Eqn. 5.123

FOV constraints during readout.

$$\Delta k_y = \frac{1}{\mathrm{FOV}_y} = \gamma \Delta \mathbf{G}_y T_{pe}$$ Eqn. 5.123

FOV constraints during phase encoding.

Field of View



$$\Delta k_x = \frac{1}{\mathrm{FOV}_x} = \gamma |\mathbf{G}_x| \Delta t$$

$$\Delta k_y = \frac{1}{\mathrm{FOV}_y} = \gamma \Delta \mathbf{G}_y T_{pe}$$

Eqn. 5.123

$$\Delta t = \frac{1}{\gamma |\mathbf{G}_x| \mathrm{FOV}_x}$$

$$\Delta \mathbf{G}_y = \frac{1}{\gamma T_{pe} \mathrm{FOV}_y}$$

Eqn. 5.124

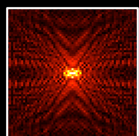
Zero Padding

Zero-Padding

- **Append zeros to *k*-space data before FFT**
  - **Append symmetrically about *k*-space**
- **Why?**
  - **If $N=2^n$, then the radix-2 FFT can be used.**
  - **Increases the "digital" resolution**
  - **Reconstruction with correct aspect ratio**

Asymmetric Resolution

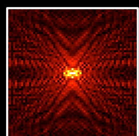
Low-Res Data



64x64

Asymmetric Resolution

Low-Res Data



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Asymmetric Resolution

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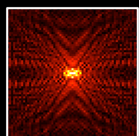
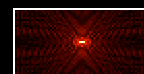


64x64

Asymmetric Res



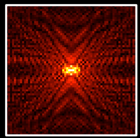
32x64

Pixels are square, but they shouldn't be.

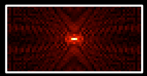
Asymmetric Resolution

Low-Res Data

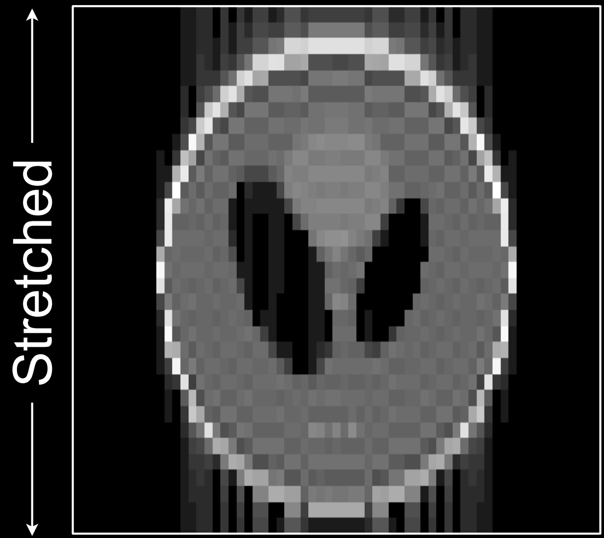
Asymmetric Res



64x64

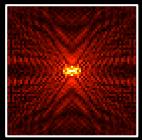


32x64



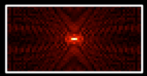
Asymmetric Resolution

Low-Res Data



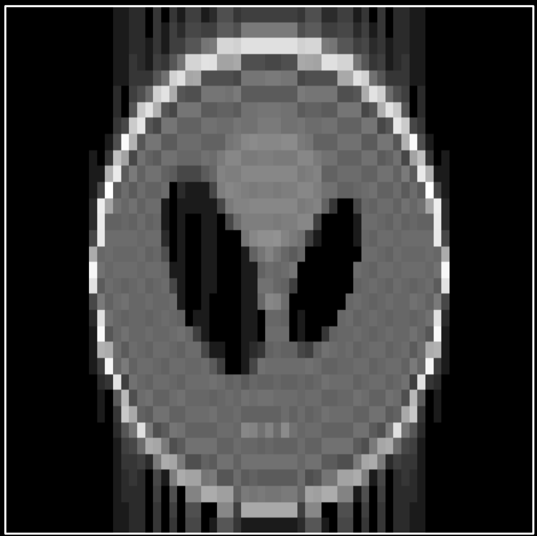
64x64

Asymmetric Res

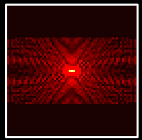


32x64

Stretched



Zero-Padded



64x"64"

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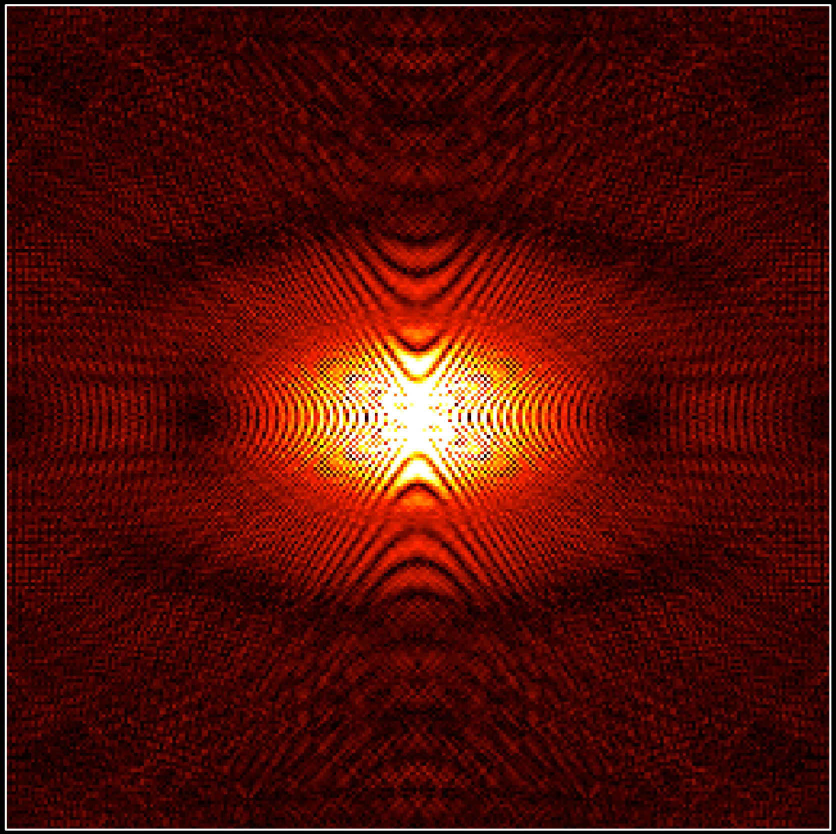
Gibb's Ringing

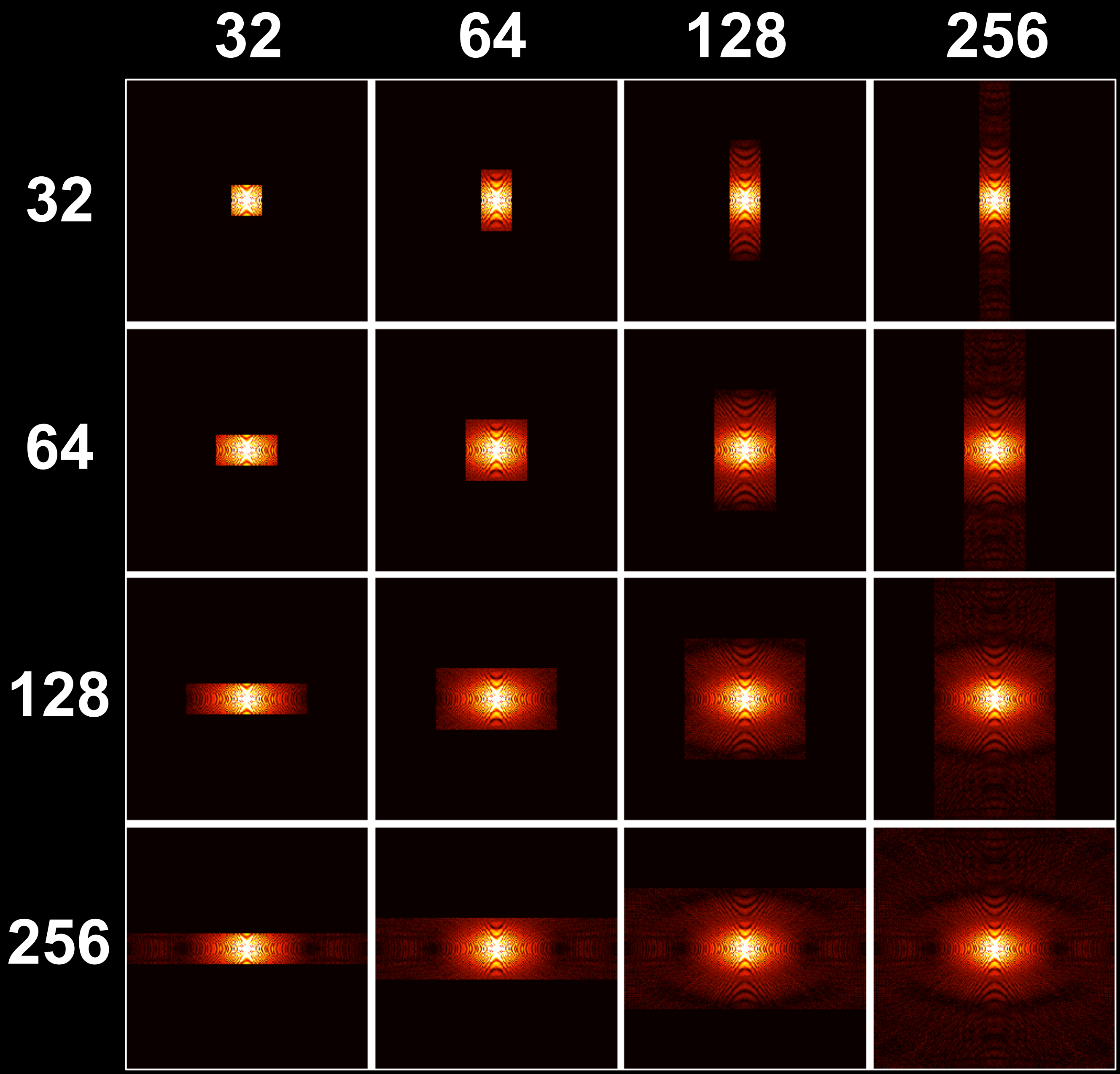
Gibb's Ringing

- Spurious ringing around sharp edges
- Max/Min overshoot is ~9% of the intensity discontinuity
  - Independent of the # of recon points
  - Frequency of ringing increases as # of recon points increases
    - Ringing becomes less apparent
- Result of truncating the Fourier series model as a consequence of finite sampling
- Can reduce by:
  - Acquiring more data
  - Filtering the data which reduces oscillations in the PSF

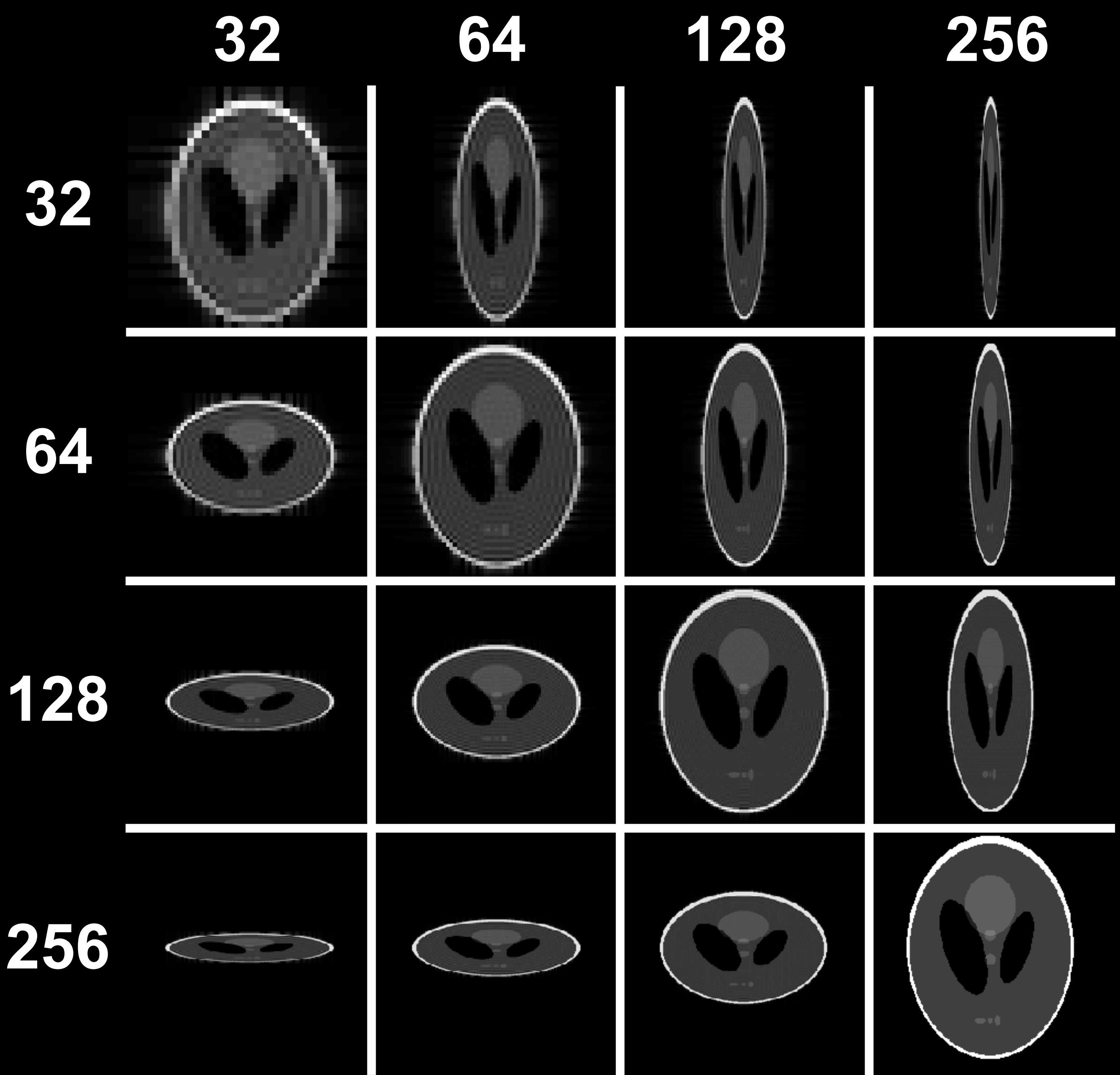
Shepp-Logan



Gibb's Ringing



Gibb's Ringing



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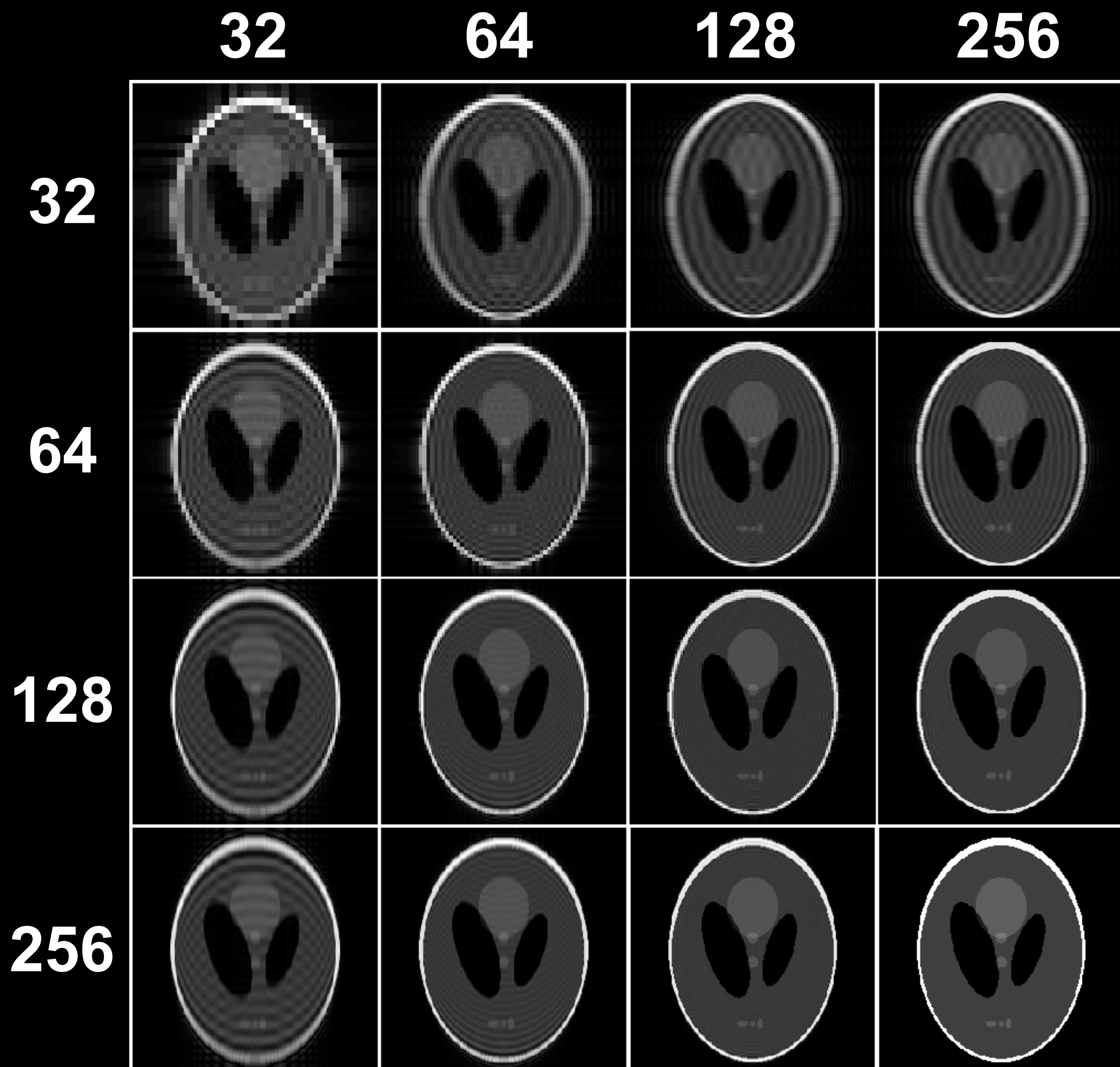
Zero-Pad



Windowed Reconstruction

Windowed Reconstruction

$$\hat{I}(x) = \Delta k \sum_{n=-N/2}^{N/2-1} S\left(n\Delta k\right) e^{i2\pi n\Delta k x}$$

Fourier reconstruction

Windowed Reconstruction

$$\hat{I}(x) = \Delta k \sum_{n=-N/2}^{N/2-1} S\left(n\Delta k\right) e^{i2\pi n\Delta k x}$$

Fourier reconstruction

$$\hat{I}(x) = \Delta k \sum_{n=-N/2}^{N/2-1} S\left(n\Delta k\right) w_n e^{i2\pi n\Delta k x}$$ Eqn. 6.21

Windowed Fourier reconstruction

k-space filter/window function

Windowed Reconstruction

$$\hat{I}(x) = I(x) * h(x)$$

Image

Object

Point Spread Function

Windowed Reconstruction

$$\hat{I}(x) = I(x) * h(x)$$

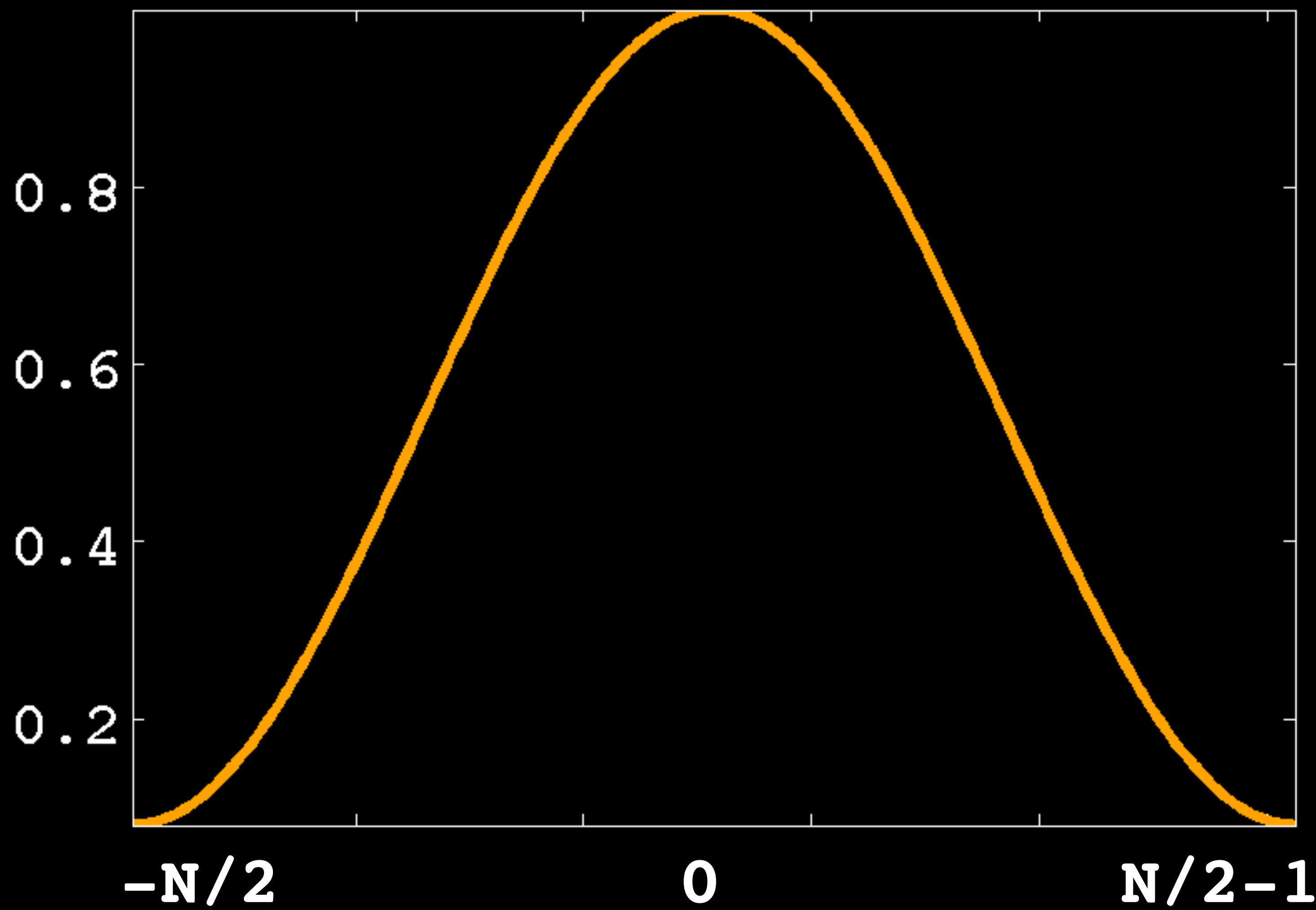
Set This To
δ-function

Point Spread Function for a windowed Fourier reconstruction.

$$h(x) = \Delta k \sum_{n=-N/2}^{N/2-1} w_n e^{i2\pi n \Delta k x}$$

Hamming Filter - 1D

$$w(n) \triangleq \begin{cases} 0.54 + 0.46\cos(2\pi\frac{n}{N}) & -N/2 \leq n \leq N/2-1 \\ 0 & \text{otherwise} \end{cases}$$



Windowed Reconstruction

FWHM PSF for a Hamming windowed Fourier reconstruction.

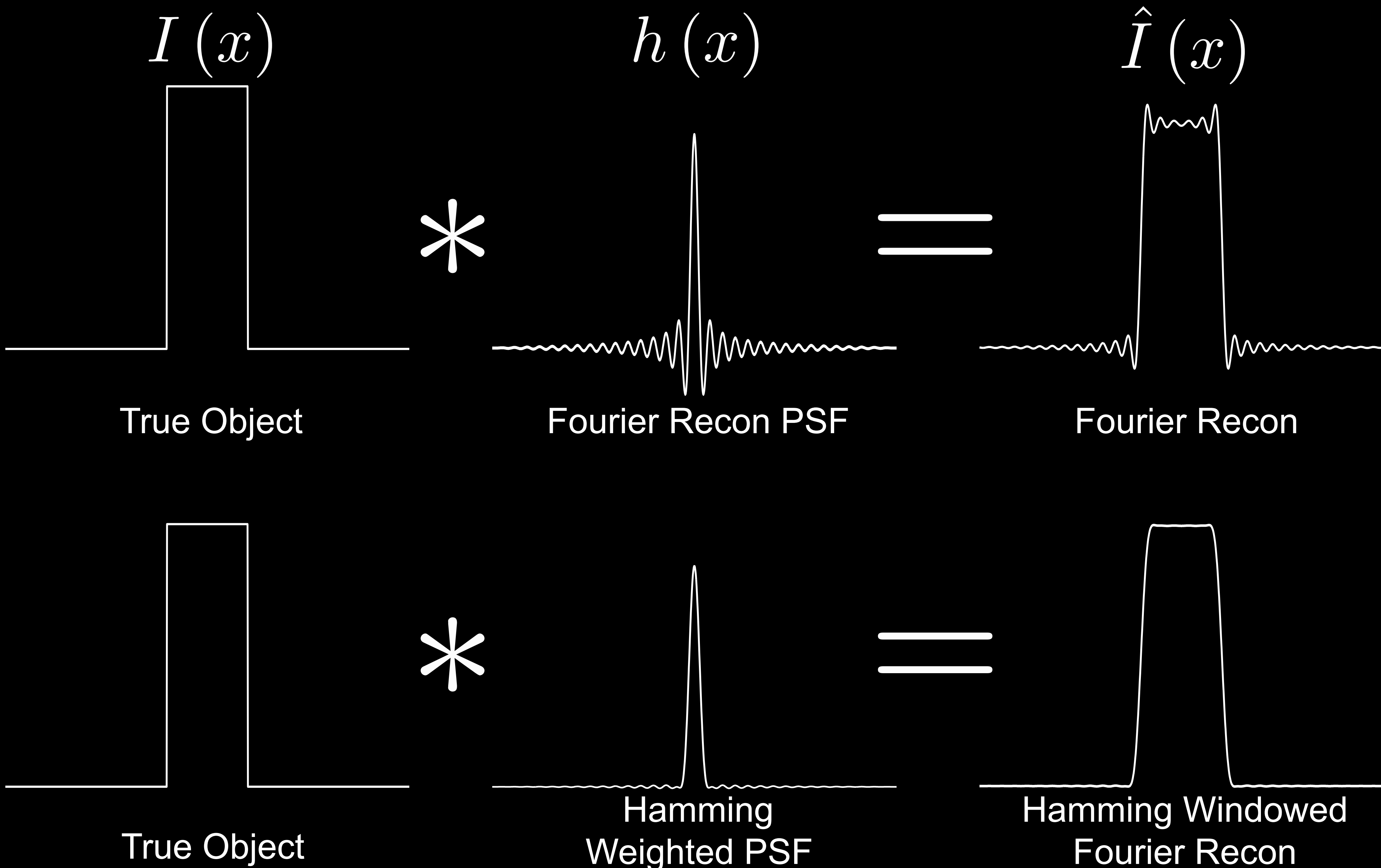
$$W_h = \left(\sum_{m=-N/2}^{N/2-1} (w_m / w_0)\, \Delta k \right)^{-1}$$

In general $w_m \leq w_0$, therefore

$$W_h \geq \frac{1}{N \Delta k}$$

Hamming windowed Fourier reconstruction suppresses ringing, but reduces effective spatial resolution.

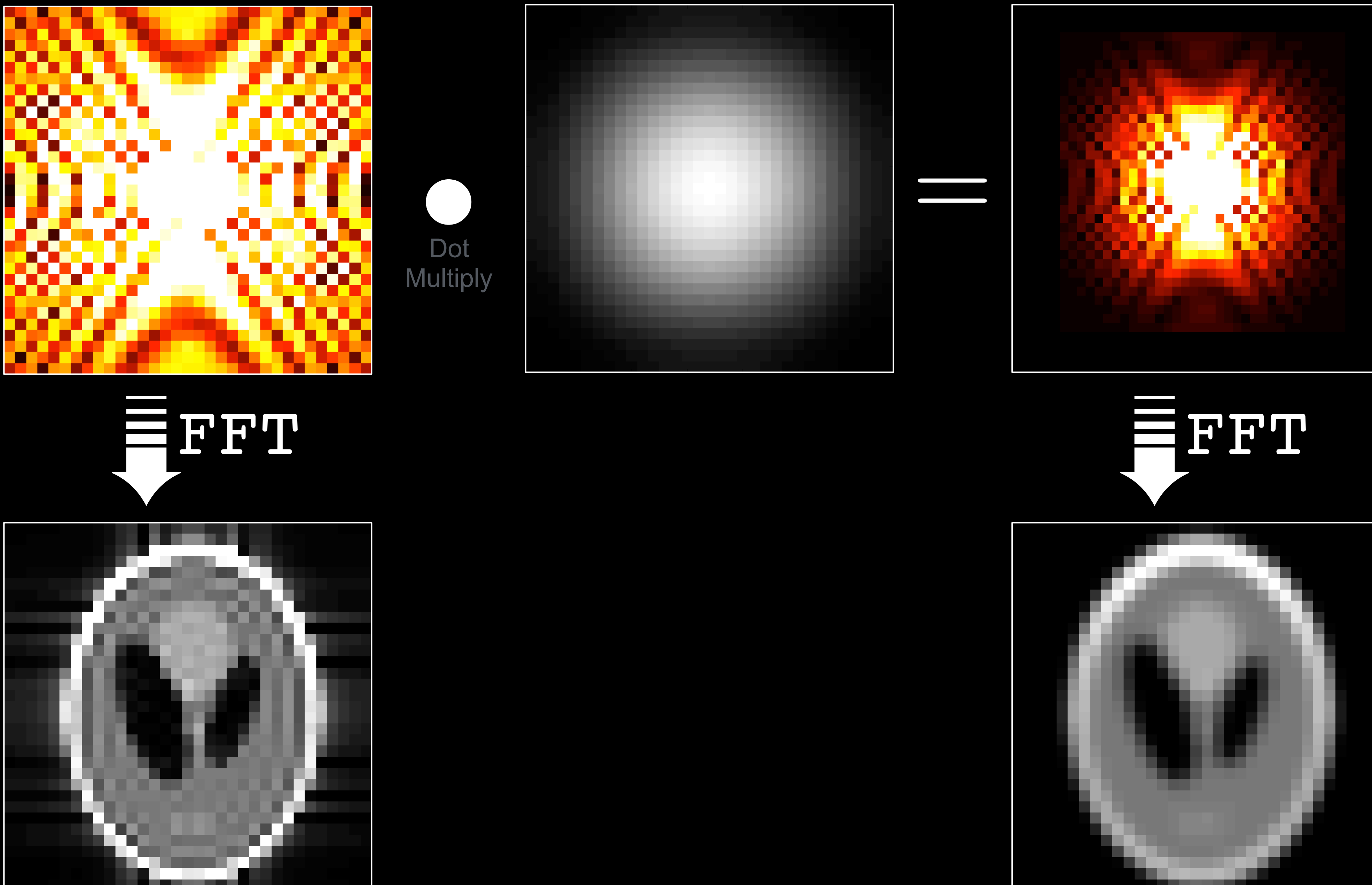
Windowed Reconstruction



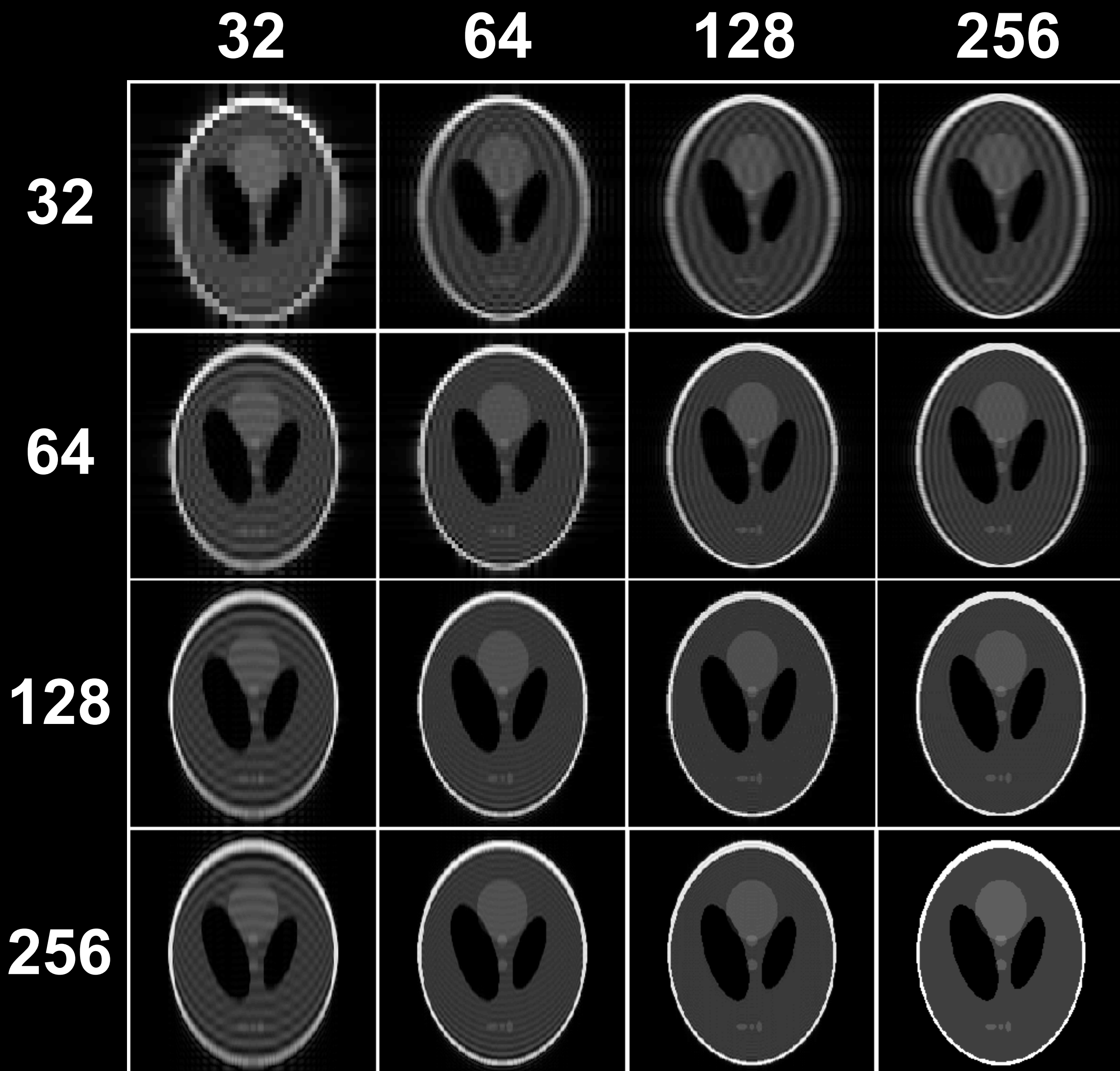
Hamming Filter - 2D

$$W(n) \triangleq w(n) \otimes w(n)$$

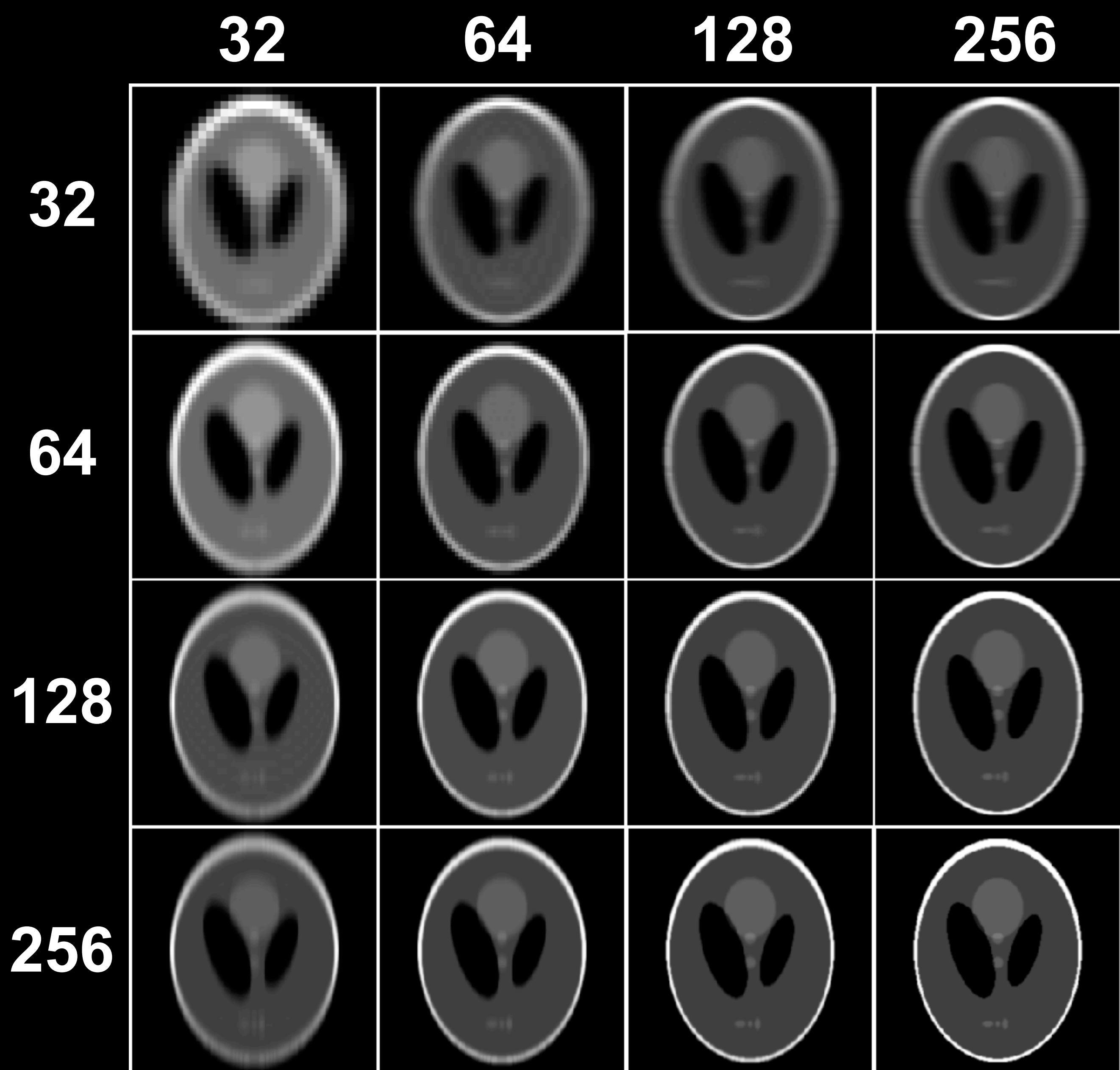
Hamming Filter



Zero-Pad



Hamming Window & Zero-Pad



Thanks

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