Bloch Equations and Relaxation

M219 - Principles and Applications of MRI

Kyung Sung, Ph.D.

1/23/2023

Course Overview

- Course website
  - https://mrrl.ucla.edu/pages/m219
- 2023 course schedule
  - https://mrrl.ucla.edu/pages/m219_2023
- Assignments
  - Homework #1 due on 1/30
  - Homework #2 will be out on 1/30
- Office hours, Fridays 10-12pm
  - In-person (Ueberroth, 1417B)
  - Zoom is also available

Last Time...

$$\vec{\tau} = \vec{\mu} \times \vec{B} \qquad \vec{S} = \vec{r} \times \vec{p}$$

$$\vec{\tau} = \frac{d\vec{S}}{dt} \qquad \vec{\mu} = \gamma \vec{S}$$

$$\frac{d\vec{\mu}}{dt} = \vec{\mu} \times \gamma \vec{B}$$

Equation of Motion for a Magnetic Dipole

$$\vec{M} = \sum_{n=1}^{N_{total}} \vec{\mu}_n$$

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$

Equation of Motion for the bulk magnetization.

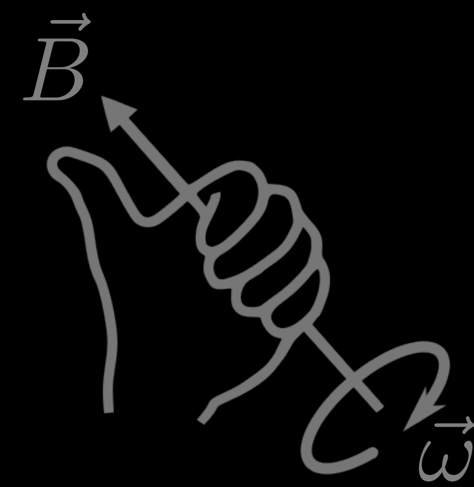
$$\vec{B}_0 = B_0 \vec{k}$$

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \left(\vec{B}_0\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ M_x & M_y & M_z \\ 0 & 0 & \gamma B_0 \end{vmatrix}$$

$$M_x(t) = M_x^0 \cos(\gamma B_0 t) + M_y^0 \sin(\gamma B_0 t)$$
$$M_y(t) = -M_x^0 \sin(\gamma B_0 t) + M_y^0 \cos(\gamma B_0 t)$$
$$M_z(t) = M_z^0$$

Free Precession w/o Relaxation

$$\mathbf{R}_z(\omega_0 t) = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t & 0 \\ -\sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Precession is left-handed (clockwise).

$$\vec{M}(t) = \mathbf{R}_z(\omega_0 t)\vec{M}^0$$

Basic RF Pulse

$$\overrightarrow{B} = \overrightarrow{B}_0 + \overrightarrow{B}_1(t)$$

$$\overrightarrow{B}_1(t) = B_1^e(t)[\cos(\omega_{RF}t + \theta)\hat{i} - \sin(\omega_{RF}t + \theta)\hat{j}]$$

$B_1^e(t)$ **pulse envelope function**

ω_{RF} **excitation carrier frequency**

θ **initial phase angle**

B_1 is perpendicular to B_0.

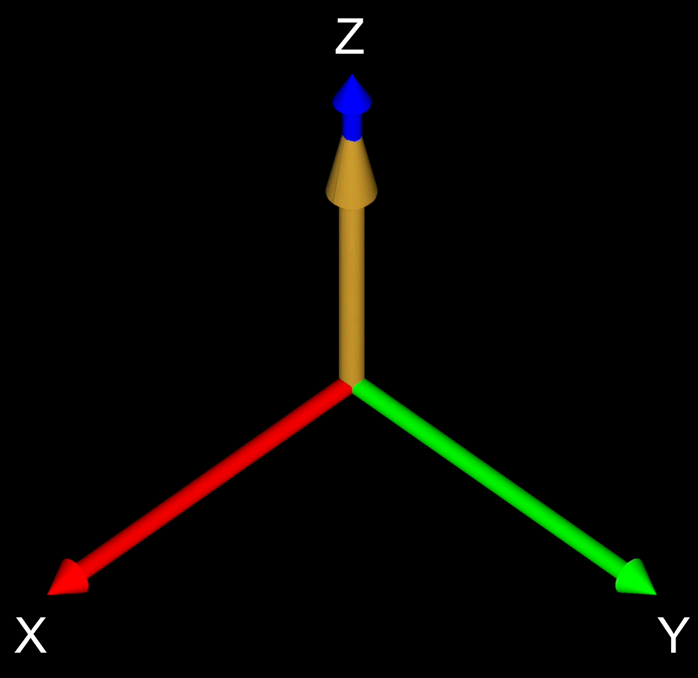
$$\overrightarrow{B}_0 = B_0\hat{k}$$

Rotating Frame

Lab vs. Rotating Frame

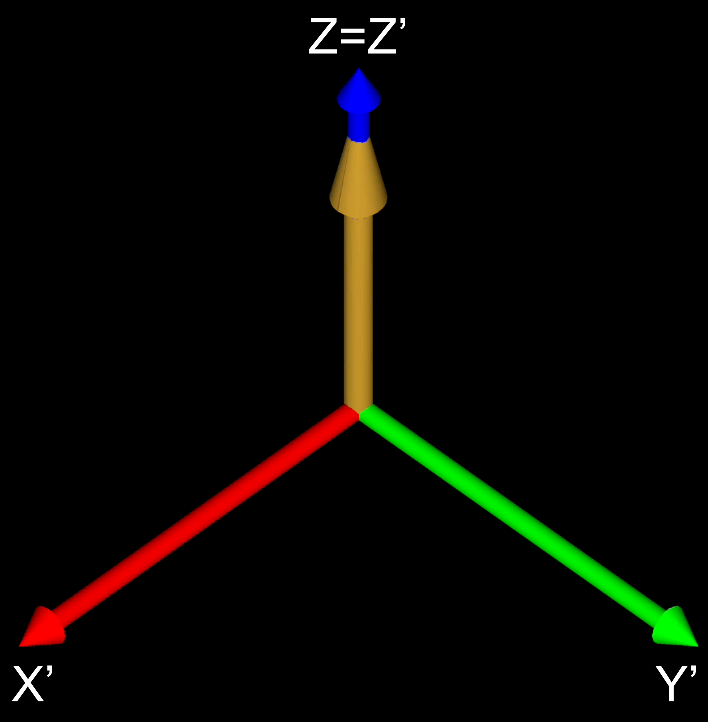
- **The rotating frame simplifies the mathematics and permits more intuitive understanding.**

90° RF (Laboratory Frame)



Spins Precess

90° RF (Rotating Frame)



Observer Precesses

Note: Both coordinate frames share the same z-axis.

Combined B_0 & B_1 Effects

$$
\begin{aligned}
\frac{d\vec{M}}{dt} &= \vec{M} \times \gamma \vec{B} \\
&= \vec{M} \times \gamma \left(\vec{B}_0 + \vec{B}_1 \right)
\end{aligned}
$$

Relationship Between Lab and Rotating Frames

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$

Rotating Frame Definitions

$$\vec{M}_{rot} \equiv \begin{bmatrix} M_{x'} \\ M_{y'} \\ M_{z'} \end{bmatrix} \qquad \vec{B}_{rot} \equiv \begin{bmatrix} B_{x'} \\ B_{y'} \\ B_{z'} \end{bmatrix} \qquad \begin{matrix} B_{z'} \equiv B_z \\ M_{z'} \equiv M_z \end{matrix}$$

$$\overrightarrow{M}_{lab}(t) = R_Z(\omega_{RF}t) \cdot \overrightarrow{M}_{rot}(t)$$

Bulk magnetization components in the rotating frame.

$$\overrightarrow{B}_{lab}(t) = R_Z(\omega_{RF}t) \cdot \overrightarrow{B}_{rot}(t)$$

Applied B-field components in the rotating frame.

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B} \quad \Rightarrow \quad \frac{d\vec{M}_{rot}}{dt} = \vec{M}_{rot} \times \gamma \vec{B}_{eff}$$

Bloch Equation (Rotating Frame)

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$

Equation of motion for an ensemble of spins (isochromats). [Laboratory Frame]

$$\frac{d\vec{M}_{rot}}{dt} = \vec{M}_{rot} \times \gamma \left(\frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot} \right)$$

Equation of motion for an ensemble of spins (isochromats). [Rotating Frame]

$$\underbrace{\vec{B}_{eff}}_{\text{Effective B-field that } M \text{ experiences in the rotating frame.}} \equiv \underbrace{\frac{\vec{\omega}_{rot}}{\gamma}}_{\text{Fictitious field that demodulates the apparent effect of } B_0.} + \underbrace{\vec{B}_{rot}}_{\text{Applied B-field in the rotating frame.}}$$

$$\vec{\omega}_{rot} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{RF} \end{pmatrix}$$

Bloch Equation (Rotating Frame)

$$\overrightarrow{B}(t) = B_0\hat{k} + B_1^e(t)[\cos(\omega_{RF}t + \theta)\hat{i} - \sin(\omega_{RF}t + \theta)\hat{j}]$$

$$\overrightarrow{B}_{lab}(t) = \begin{pmatrix} B_1^e(t)\cos(\omega_{RF}t + \theta) \\ -B_1^e(t)\sin(\omega_{RF} + \theta) \\ B_0 \end{pmatrix} \qquad \overrightarrow{B}_{rot}(t) = \begin{pmatrix} B_1^e(t)\cos\theta \\ -B_1^e(t)\sin\theta \\ B_0 \end{pmatrix}$$

$$\vec{B}_{eff} \equiv \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot} \qquad \overrightarrow{\omega}_{rot} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{RF} \end{pmatrix}$$

$\vec{B}_{eff}$: Effective B-field that *M* experiences in the rotating frame.

$\frac{\vec{\omega}_{rot}}{\gamma}$: Fictitious field that demodulates the apparent effect of B_0.

$\vec{B}_{rot}$: Applied B-field in the rotating frame.

Bloch Equation (Rotating Frame)

$$\vec{B}_{eff} \equiv \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot}$$

Assume no RF phase (θ = 0)

$$\overrightarrow{B}_{rot}(t) = \begin{pmatrix} B_1^e(t) \\ 0 \\ B_0 \end{pmatrix} \qquad \overrightarrow{\omega}_{rot} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{RF} \end{pmatrix}$$

$$\overrightarrow{B}_{eff}(t) = \begin{pmatrix} B_1^e(t) \\ 0 \\ \cancel{B_0 - \frac{\omega_{RF}}{\gamma}} \end{pmatrix}$$

Relationship Between Lab and Rotating Frames

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$

Rotating Frame Definitions

$$\vec{M}_{rot} \equiv \begin{bmatrix} M_{x'} \\ M_{y'} \\ M_{z'} \end{bmatrix} \qquad \vec{B}_{rot} \equiv \begin{bmatrix} B_{x'} \\ B_{y'} \\ B_{z'} \end{bmatrix} \qquad \begin{array}{c} B_{z'} \equiv B_z \\ M_{z'} \equiv M_z \end{array}$$

$$\overrightarrow{M}_{lab}(t) = R_Z(\omega_{RF}t) \cdot \overrightarrow{M}_{rot}(t)$$

Bulk magnetization components in the rotating frame.

$$\overrightarrow{B}_{lab}(t) = R_Z(\omega_{RF}t) \cdot \overrightarrow{B}_{rot}(t)$$

Applied B-field components in the rotating frame.

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B} \quad \Rightarrow \quad \frac{d\vec{M}_{rot}}{dt} = \vec{M}_{rot} \times \gamma \vec{B}_{eff}$$

Bloch Equation (Rotating Frame)

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$

Equation of motion for an ensemble of spins (isochromats). [Laboratory Frame]

$$\frac{d\vec{M}_{rot}}{dt} = \vec{M}_{rot} \times \gamma \left(\frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot} \right)$$

Equation of motion for an ensemble of spins (isochromats). [Rotating Frame]

$$\vec{B}_{eff} \equiv \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot}$$

$$\vec{\omega}_{rot} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{RF} \end{pmatrix}$$

Effective B-field that *M* experiences in the rotating frame.

Fictitious field that demodulates the apparent effect of B_0.

Applied B-field in the rotating frame.

Bloch Equation (Rotating Frame)

$$\overrightarrow{B}(t) = B_0\hat{k} + B_1^e(t)[\cos(\omega_{RF}t + \theta)\hat{i} - \sin(\omega_{RF}t + \theta)\hat{j}]$$

$$\overrightarrow{B}_{lab}(t) = \begin{pmatrix} B_1^e(t)\cos(\omega_{RF}t + \theta) \\ -B_1^e(t)\sin(\omega_{RF} + \theta) \\ B_0 \end{pmatrix} \qquad \overrightarrow{B}_{rot}(t) = \begin{pmatrix} B_1^e(t)\cos\theta \\ -B_1^e(t)\sin\theta \\ B_0 \end{pmatrix}$$

$$\vec{B}_{eff} \equiv \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot} \qquad \overrightarrow{\omega}_{rot} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{RF} \end{pmatrix}$$

Effective B-field that *M* experiences in the rotating frame.

Fictitious field that demodulates the apparent effect of B_0.

Applied B-field in the rotating frame.

Bloch Equation (Rotating Frame)

$$\vec{B}_{eff} \equiv \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot}$$

Assume no RF phase (θ = 0)

$$\overrightarrow{B}_{rot}(t) = \begin{pmatrix} B_1^e(t) \\ 0 \\ B_0 \end{pmatrix} \qquad \overrightarrow{\omega}_{rot} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{RF} \end{pmatrix}$$

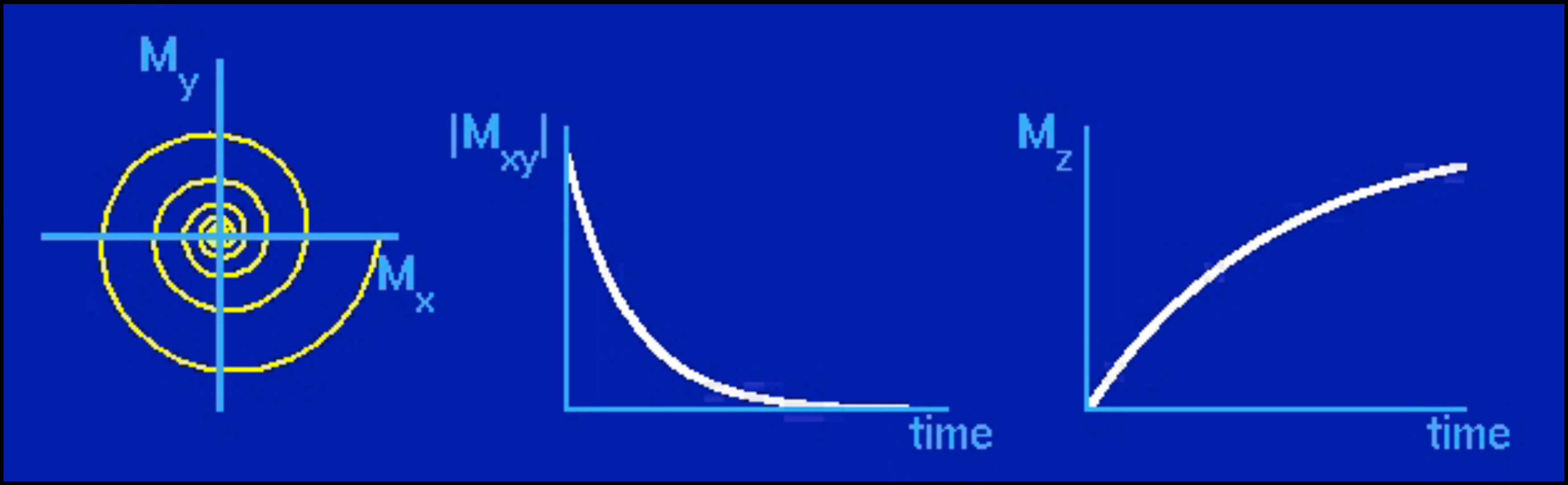
$$\overrightarrow{B}_{eff}(t) = \begin{pmatrix} B_1^e(t) \\ 0 \\ B_0 - \frac{\omega_{RF}}{\gamma} \end{pmatrix}$$

To the Board

T_1 & T_2 Relaxation

Relaxation

- Magnetization returns exponentially to equilibrium:
  - Longitudinal recovery time constant is T1
  - Transverse decay time constant is T2
- Relaxation and precession are independent



T_1 Relaxation

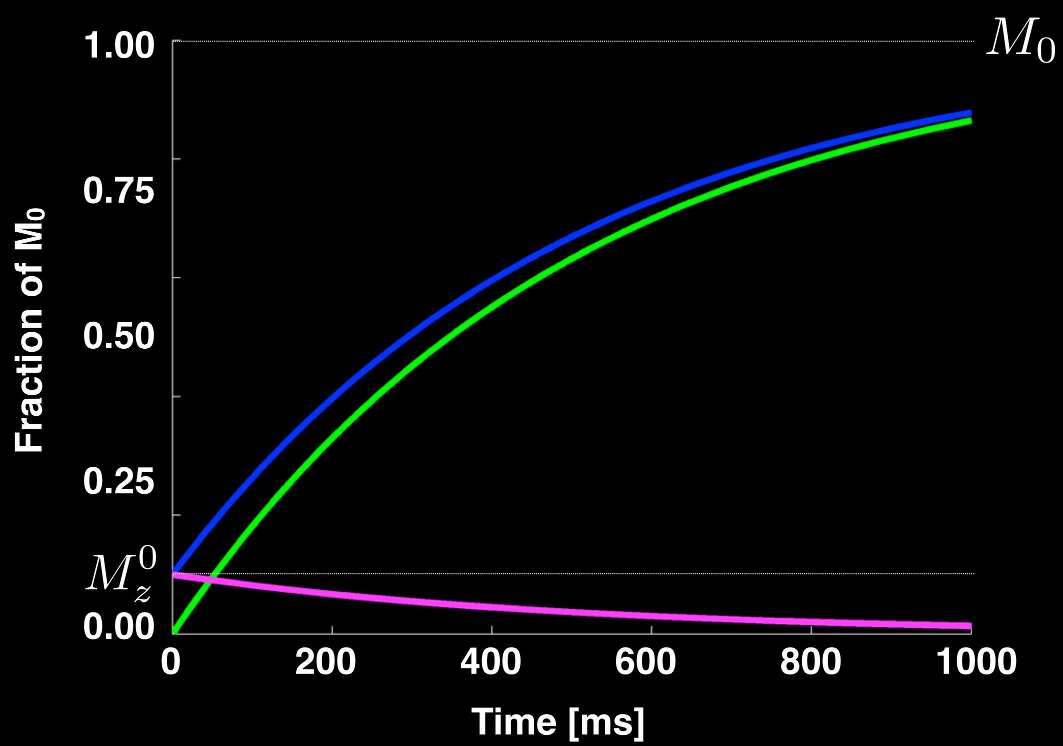
- Longitudinal or spin-lattice relaxation
  - Typically, (10s ms) < T1 < (100s ms)
- T1 is long for
  - Small molecules (water)
  - Large molecules (proteins)
- T1 is short for
  - Fats and intermediate-sized molecules
- T1 increases with increasing B0
- T1 decreases with contrast agents

Short T_1s are bright on T_1-weighted image

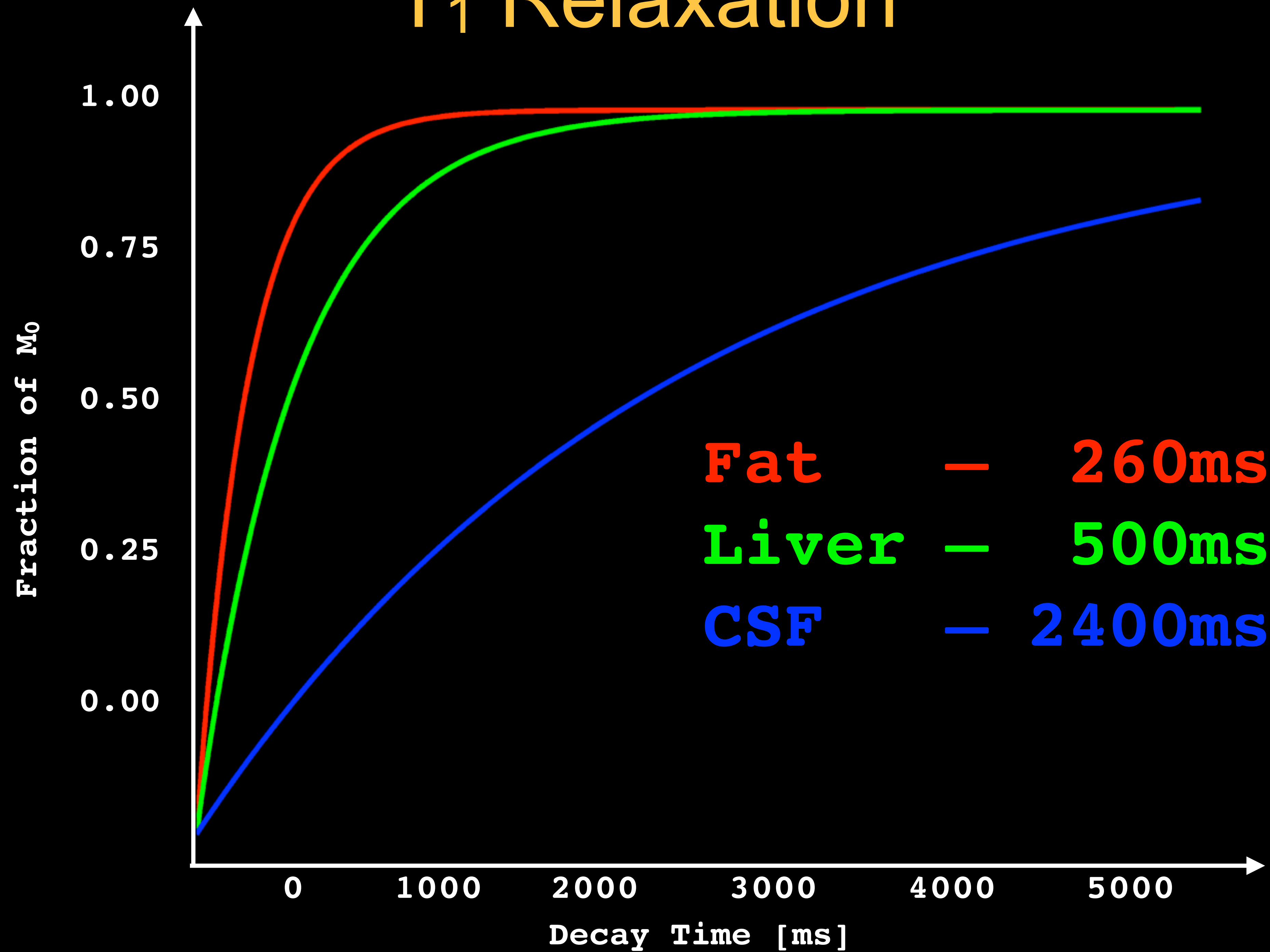
T_1 Relaxation

Free Precession in the Lab ***or*** Rotating Frame with Relaxation

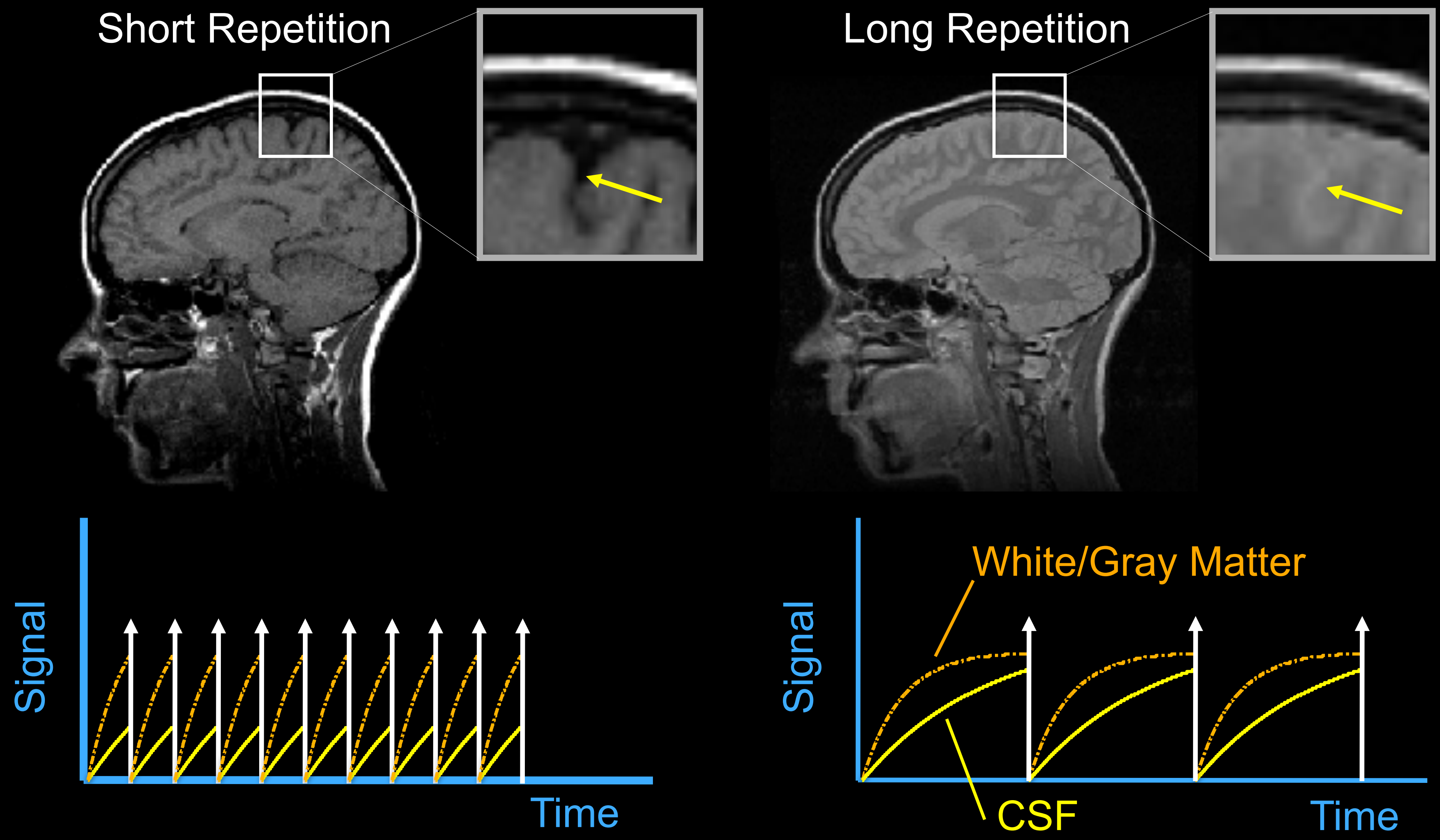
$$\underbrace{M_{z'}(t)}_{\text{Net Magnetization}} = \underbrace{M_z^0 e^{-t/T_1}}_{\text{Prepared Magnetization Decays } (M_z^0)} + \underbrace{M_0\left(1 - e^{-t/T_1}\right)}_{\text{Return to Thermal Equilibrium } (M_0)}$$



T_1 Relaxation



T_1 Contrast



Short Repetition
Long Repetition
Signal
Time
Signal
White/Gray Matter
CSF
Time

T_2 Relaxation

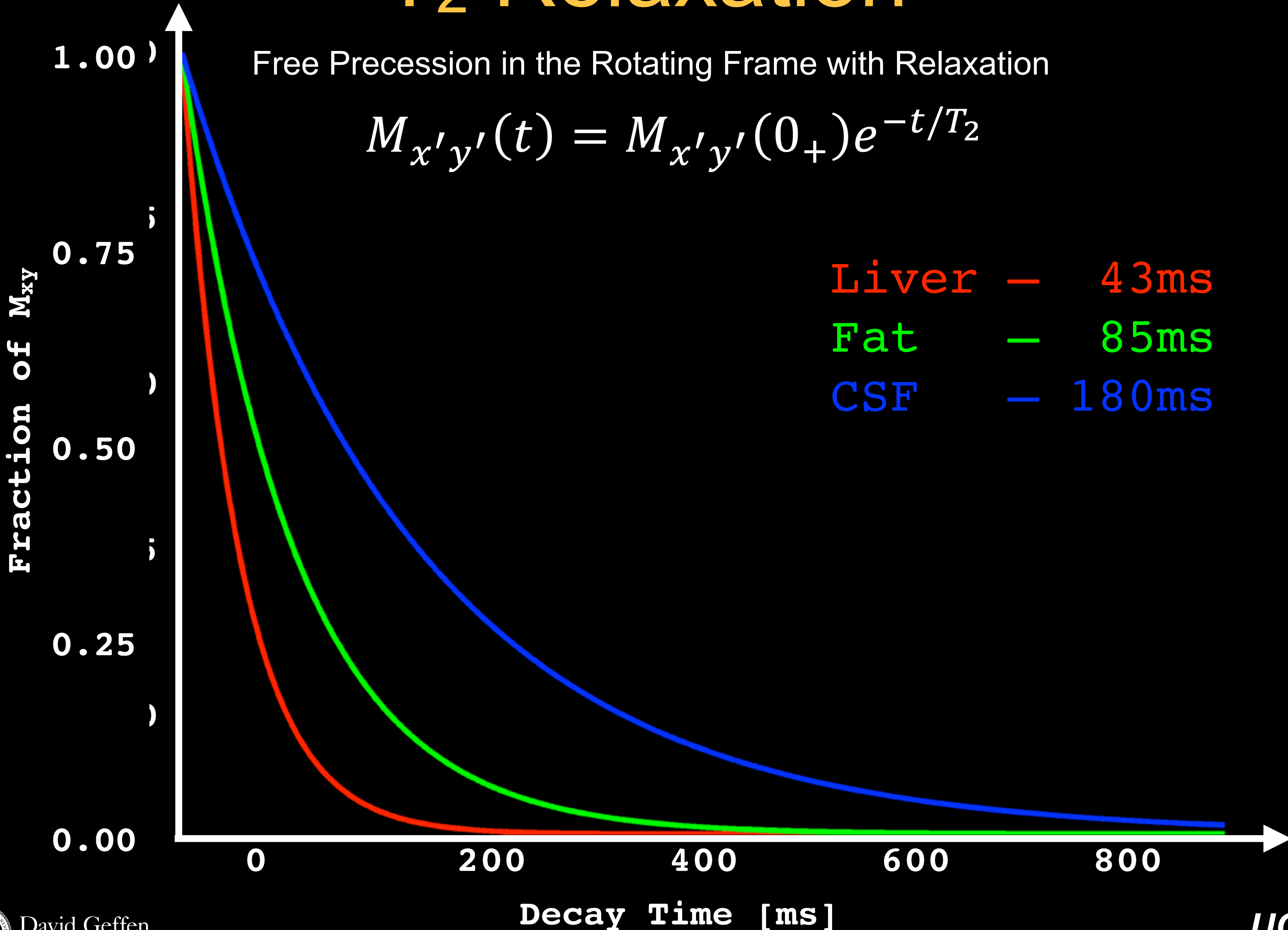
- Transverse or spin-spin relaxation
  - Molecular interaction causes spin dephasing
  - Typically, T2 < (10s ms)
- Increasing molecular size, decrease T2
  - Fat has a short T2
- Increasing molecular mobility, increases T2
  - Liquids (CSF, edema) have long T2s
- Increasing molecular interactions, decreases T2
  - Solids have short T2s
- T2 relatively independent of B0

Long T_2 is bright on T_2 weighted image

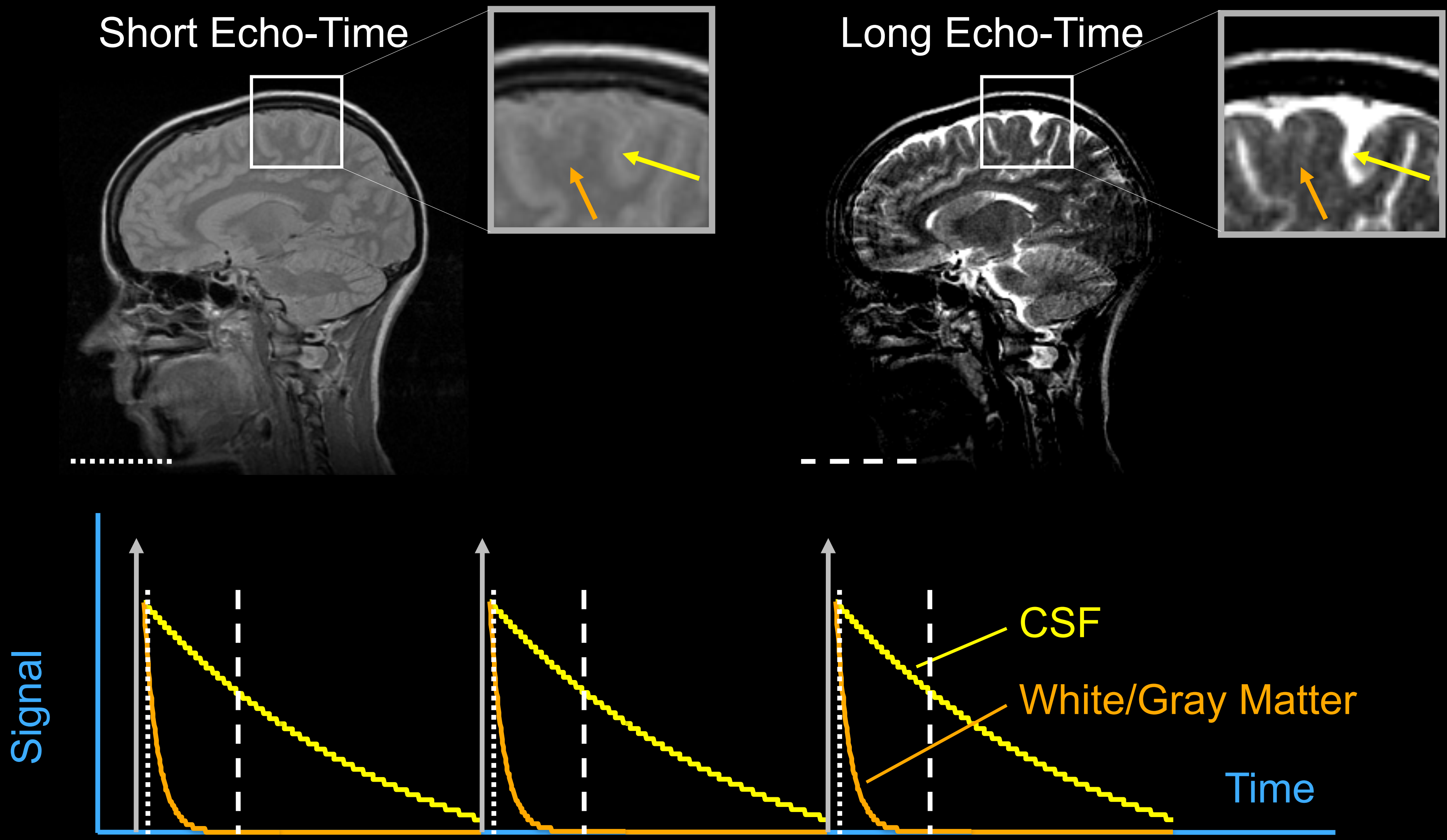
T_2 Relaxation

Free Precession in the Rotating Frame with Relaxation

$$M_{x'y'}(t) = M_{x'y'}(0_+)e^{-t/T_2}$$



T2 Contrast



Short Echo-Time
Long Echo-Time
CSF
White/Gray Matter
Signal
Time

T_1 and T_2 Values @ 1.5T

Tissue	T_1 [ms]	T_2 [ms]
gray matter	925	100
white matter	790	92
muscle	875	47
fat	260	85
kidney	650	58
liver	500	43
CSF	2400	180

Each tissue has “unique” relaxation properties, which enables “soft tissue contrast”.

T_2^* Relaxation

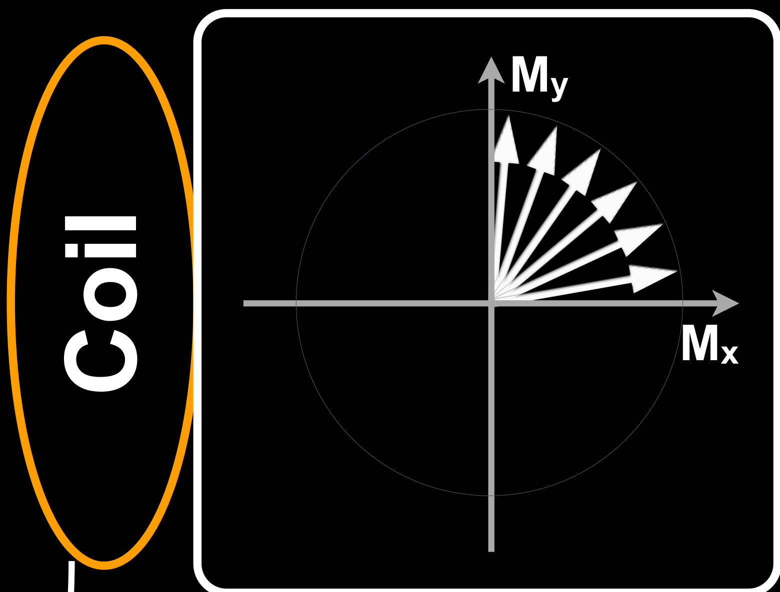
T_2^* Relaxation

$$\frac{1}{T_2^*} = \frac{1}{T_2} + \gamma \Delta B_0$$

- T_2* is "observed" transverse relaxation time constant
- T_2* consists of *irreversible spin-spin (T_2) dephasing* and *reversible intravoxel spin de-phasing* due to off-resonance
- Sources of off-resonance:
  - B_0 inhomogeneity
  - susceptibility differences (e.g. air spaces)

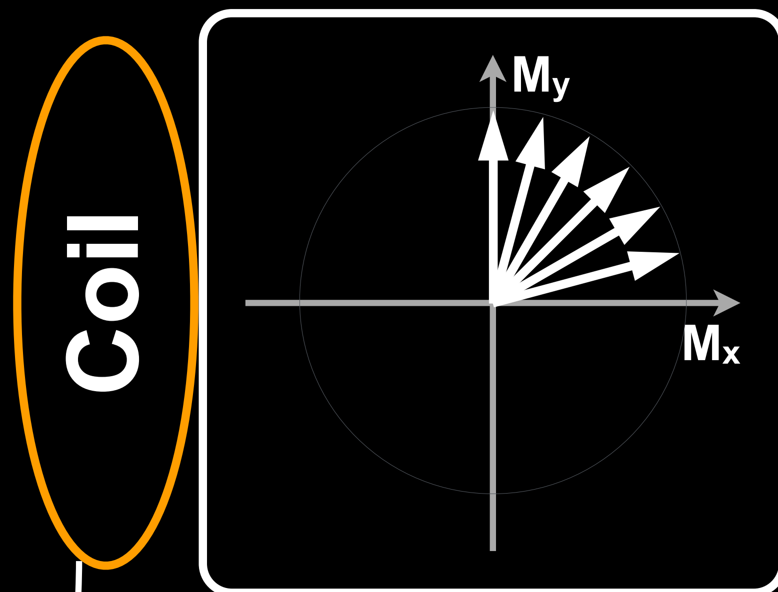
T_2 versus T_2*

T_2 Decay



Signal loss from spin-spin interaction.

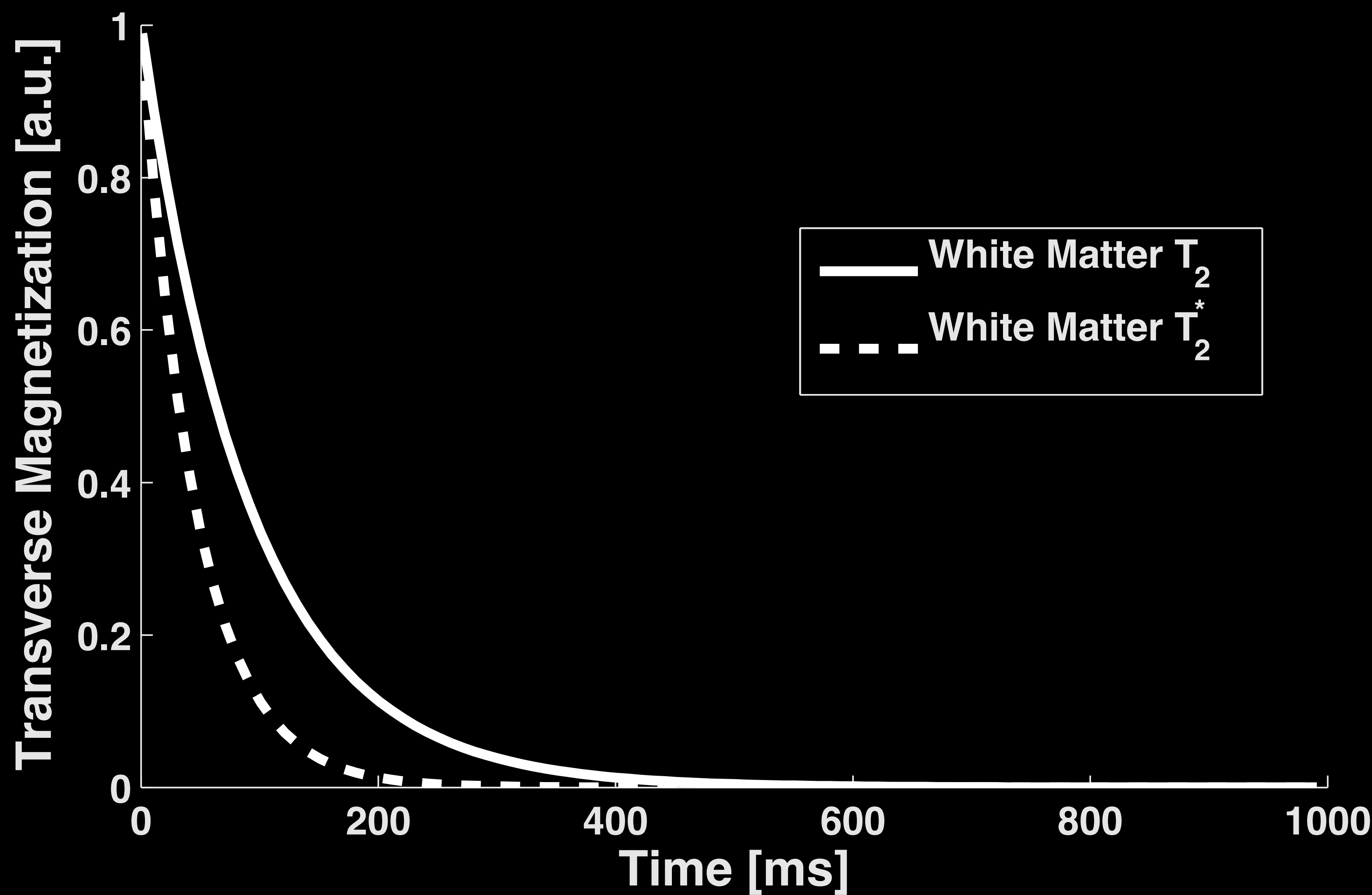
T_2* Decay



Signal loss from spin-spin interaction and off-resonance dephasing and T_2*.

T_2* is signal loss from spin dephasing and T_2

T2*<T2 (always!)


Transverse Magnetization [a.u.]
1
0.8
0.6
0.4
0.2
0
0
200
400
600
800
1000
Time [ms]
White Matter T_2
White Matter T_2^*

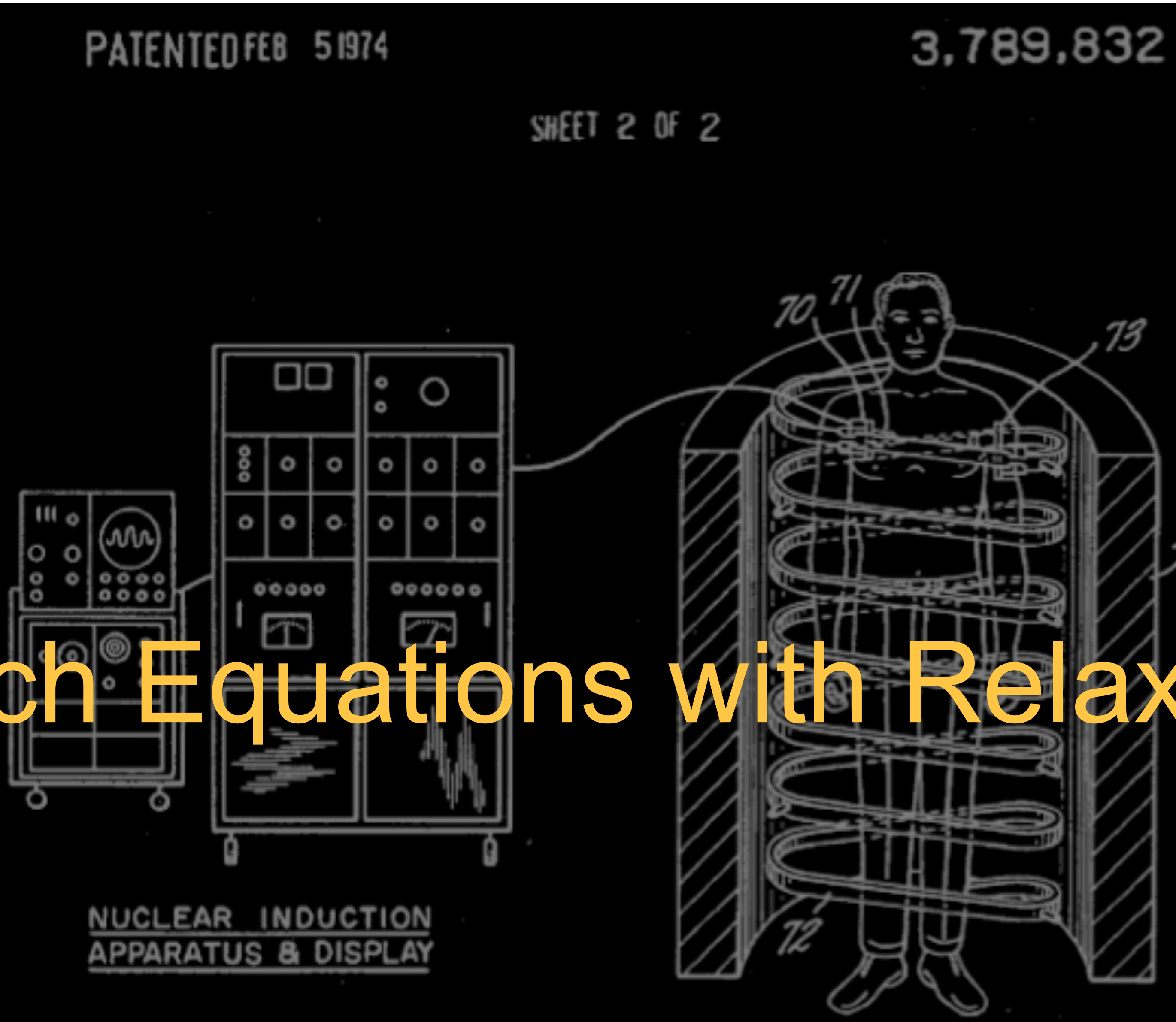


FIG. 2

Bloch Equations with Relaxation

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Bloch Equations with Relaxation

$$\frac{d\vec{\mathrm{M}}}{dt} = \vec{\mathrm{M}} \times \gamma \vec{\mathrm{B}} - \frac{M_x \hat{\mathrm{i}} + M_y \hat{\mathrm{j}}}{T_2} - \frac{(M_z - M_0)\,\hat{\mathrm{k}}}{T_1}$$

- **Differential Equation**
  - Ordinary, Coupled, Non-linear
- **No analytic solution, in general.**
  - Analytic solutions for simple cases.
  - Numerical solutions for all cases.
- **Phenomenological**
  - Exponential behavior is an approximation.

Bloch Equations - Lab Frame

$$\frac{d\vec{\mathrm{M}}}{dt} = \underbrace{\vec{\mathrm{M}} \times \gamma \vec{\mathrm{B}}}_{\text{Precession}} - \underbrace{\frac{M_x \hat{\mathrm{i}} + M_y \hat{\mathrm{j}}}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_z - M_0)\,\hat{\mathrm{k}}}{T_1}}_{\text{Longitudinal Relaxation}}$$

- Precession
  - Magnitude of M unchanged
  - Phase (rotation) of M changes due to B
- Relaxation
  - T_1 changes are slow O(100ms)
  - T_2 changes are fast O(10ms)
  - Magnitude of M can be ZERO
- Diffusion
  - Spins are thermodynamically driven to exchange positions.
  - Bloch-Torrey Equations

Bloch Equations – Rotating Frame

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \underbrace{\gamma \vec{M}_{rot} \times \vec{B}_{eff}}_{\text{“Precession”}} - \underbrace{\frac{M_{x'}\vec{i}' + M_{y'}\vec{j}'}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_{z'} - M_0)\vec{k}'}{T_1}}_{\text{Longitudinal Relaxation}}$$

$$\vec{B}_{eff} \triangleq \frac{\vec{\omega}}{\gamma} + \vec{B}_{rot}$$

The applied B_0 and B_1 field in the rotating frame

Effective B-field that M experiences in the rotating frame

Fictitious field created by the rotating frame that demodulates the apparent effect of B_0

Free Precession in the Rotating Frame with Relaxation

Free Precession in the Rotating Frame

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \gamma \vec{M}_{rot} \times \vec{B}_{eff} - \frac{M_{x'}\vec{i}' + M_{y'}\vec{j}'}{T_2} - \frac{(M_{z'} - M_0)\vec{k}'}{T_1}$$

$$\vec{B}_{eff} \triangleq \frac{\vec{\omega}}{\gamma} + \vec{B}_{rot}$$

$$\vec{\omega}_{rot} = \vec{\omega} = -\gamma \mathrm{B}_0 \hat{k} \qquad \vec{\mathrm{B}}_{rot} = \mathrm{B}_0 \hat{k}$$

$$\vec{B}_{eff} = \vec{0}$$

$$\frac{\partial \vec{M}_{rot}}{\partial t} = -\frac{M_{x'}\vec{i}' + M_{y'}\vec{j}'}{T_2} - \frac{(M_{z'} - M_0)\vec{k}'}{T_1}$$

The precessional term drops out in the rotating frame.

Free Precession in the Rotating Frame

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \underbrace{-\frac{M_{x'}\vec{i}' + M_{y'}\vec{j}'}{T_2}}_{\text{Transverse Relaxation}} \underbrace{-\frac{(M_{z'} - M_0)\vec{k}'}{T_1}}_{\text{Longitudinal Relaxation}}$$

- **No precession**
- **T_1 and T_2 Relaxation**
- **Drop the diffusion term**
- **System or first order, linear, separable ODEs!**

The precessional term drops out in the rotating frame.

Free Precession in the Rotating Frame

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \underbrace{-\frac{M_{x'}\vec{i}' + M_{y'}\vec{j}'}{T_2}}_{\text{Transverse Relaxation}} \underbrace{-\frac{(M_{z'} - M_0)\vec{k}'}{T_1}}_{\text{Longitudinal Relaxation}}$$

Solution:

$$M_{z'}(t) = M_z^0 e^{-t/T_1} + M_0\left(1 - e^{-t/T_1}\right)$$

$$M_{x'y'}(t) = M_{x'y'}(0_+)e^{-t/T_2}$$

The precessional term drops out in the rotating frame.

Forced Precession in the Rotating Frame with Relaxation

Forced Precession in the Rot. Frame with Relaxation

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \gamma \vec{M}_{rot} \times \vec{B}_{eff} - \frac{M_{x'}\vec{i}' + M_{y'}\vec{j}'}{T_2} - \frac{(M_{z'} - M_0)\vec{k}'}{T_1}$$

$$\vec{B}_{eff} \triangleq \frac{\vec{\omega}}{\gamma} + \vec{B}_{rot}$$

$$\vec{\omega}_{rot} = \vec{\omega} = -\gamma \mathrm{B}_0 \hat{k} \qquad \vec{B}_{rot} = B_0 \hat{k} + B_1^e(t)\hat{i}'$$

$$\vec{B}_{eff} = B_1^e(t)\hat{i}'$$

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The precessional term ***does not*** drop out in the rotating frame.

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Forced Precession in the Rot. Frame with Relaxation

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \gamma \vec{M}_{rot} \times \vec{B}_{eff} - \frac{M_{x'}\vec{i}' + M_{y'}\vec{j}'}{T_2} - \frac{(M_{z'} - M_0)\vec{k}'}{T_1}$$

$$\vec{B}_{eff} = B_1^e(t)\hat{\vec{i}}'$$

- **B1 induced nutation**
- **T_1 and T_2 Relaxation**
- **Drop the diffusion term**
- **System or first order, linear, coupled PDEs!**
- **When does this equation apply?**

The precessional term ***does not*** drop out in the rotating frame.

Forced Precession in the Rotating Frame with Relaxation

- RF pulses are short
  - 100μs to 5ms
- Relaxation time constants are long
  - T_1 O(100s) ms
  - T_2 O(10s) ms
- Complicated Coupling
- Best suited for simulation

Free? Forced? Relaxation?

- We've considered all combinations of:
  - Free and forced precession
  - With and without relaxation
  - Laboratory and rotating frames
- Which one's concern M219 the most?
  - Free precession in the rotating frame with relaxation
  - Forced precession in the rotating frame without relaxation.
- We can, in fact, simulate all of them…

Questions?

- Related reading materials
  - Nishimura - Chap 4 and 5

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