Compressed Sensing MRI

M229 Advanced Topics in MRI

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4/29/2025

Today's topics

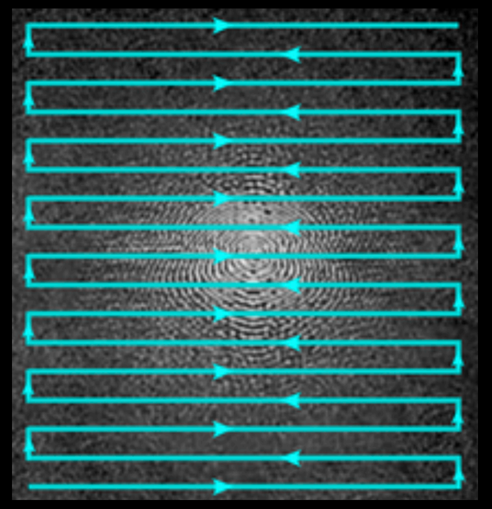
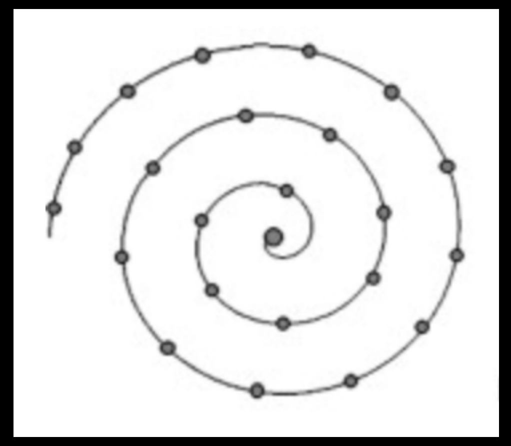
- k-Space properties review
- Compressed sensing MRI (with code examples)
    - Sparse representation
    - Incoherent artifacts
    - Nonlinear reconstruction
- Compressed sensing MRI applications

MRI acceleration

- MRI acquisition time is limited by
  - MRI physics (encoding mechanisms, relaxation properties…)
  - Hardware constraints (gradient switching…)

- Shorter MRI scan time can
  - Improve patient comfort
  - Reduce occurrence of motion artifacts
  - (from a hospital's view) Increase throughput with better resource management

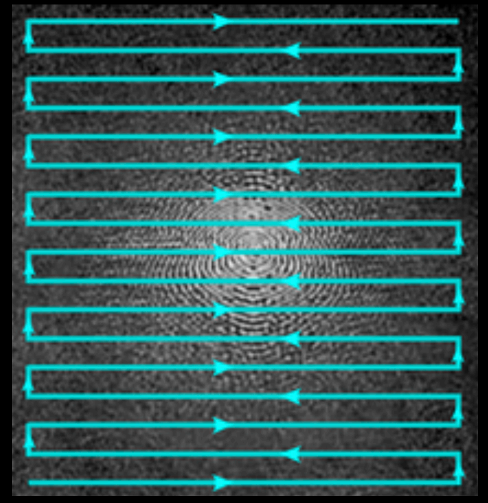
MRI acceleration

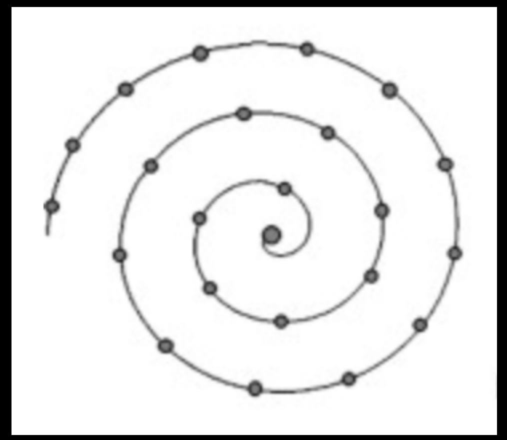



- Different acceleration strategies:
  - (1) Sequence design: Using a rapid acquisition strategy (e.g., EPI and spiral imaging)
  - (2) Simultaneous multi-slice (SMS) techniques: Using specialized RF pulses to excite multiple slices at the same time, followed by advanced reconstruction
  - (3) Data undersampling with advanced reconstruction:
    - a. Partial Fourier reconstruction: Using conjugate symmetry in k-space
    - b. Parallel imaging: Using sensitivity information from multiple coils
    - c. Compressed sensing: Using sparsity constraints for reconstruction
    - d. Deep learning: Using non-linear neural network trained with large datasets
  - (4) Hybrid techniques: Combining different acceleration strategies to achieve robust acceleration

https://mriquestions.com/echo-planar-imaging.html

MRI acceleration





- Different acceleration strategies:
  - (1) Sequence design: Using a rapid acquisition strategy (e.g., EPI and spiral imaging)

    Will be covered on 5/8 - 5/20

  - (2) Simultaneous multi-slice (SMS) techniques: Using specialized RF pulses to excite multiple slices at the same time, followed by advanced reconstruction

  - (3) Data undersampling with advanced reconstruction:
    - a. Partial Fourier reconstruction: Using conjugate symmetry in k-space — Already covered on 4/22
    - b. Parallel imaging: Using sensitivity information from multiple coils — Already covered on 4/24
    - c. Compressed sensing: Using sparsity constraints for reconstruction — TODAY!
    - d. Deep learning: Using non-linear neural network trained with large datasets — Will be covered on 5/22

  - (4) Hybrid techniques: Combining different acceleration strategies to achieve robust acceleration

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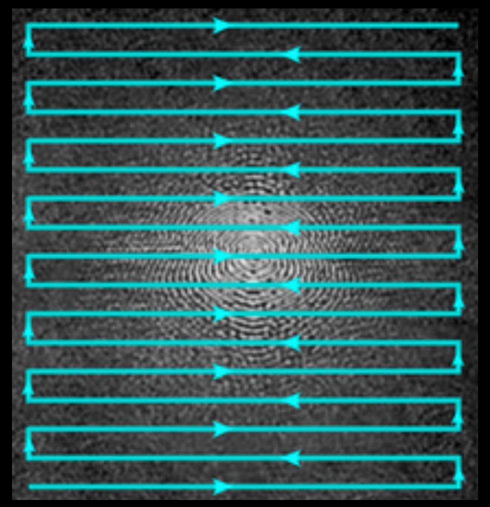
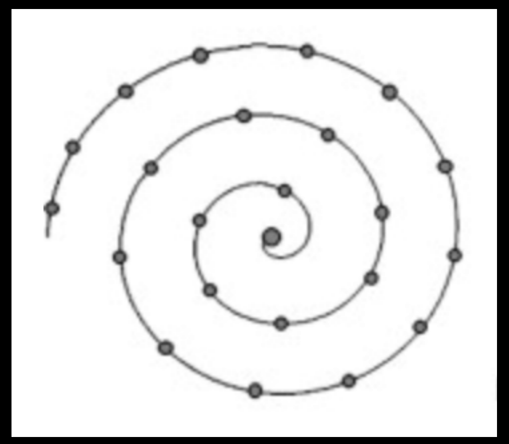
MRI acceleration

- Different acceleration strategies:
  - (1) Sequence design: Using a rapid acquisition strategy (e.g., EPI and spiral imaging)
  - (2) Simultaneous multi-slice (SMS) techniques: Using specialized RF pulses to excite multiple slices at the same time, followed by advanced reconstruction
  - (3) Data undersampling with advanced reconstruction:

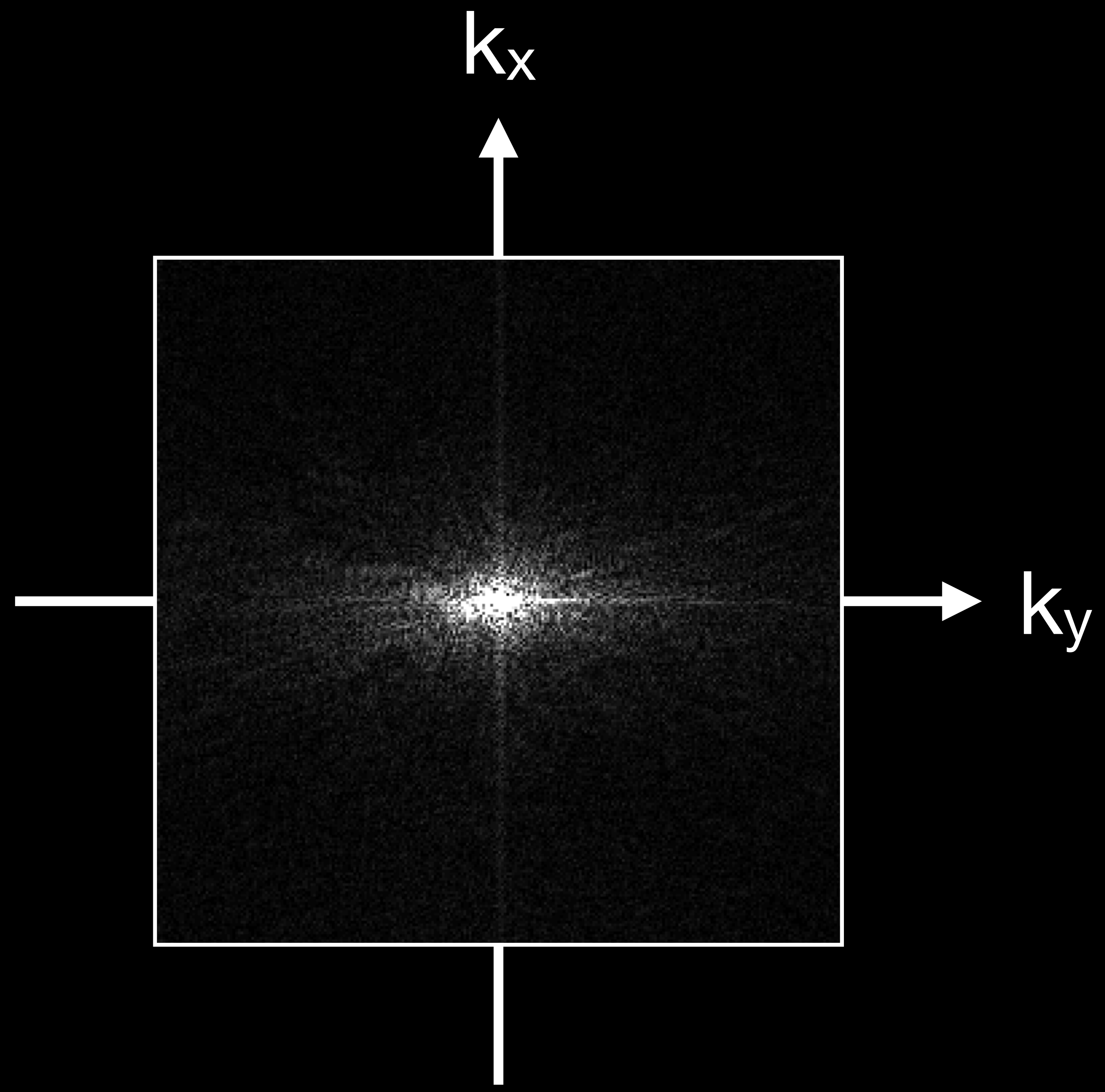
    These techniques all require some prior information for image reconstruction

    - a. Partial Fourier reconstruction: Using conjugate symmetry in k-space
    - b. Parallel imaging: Using sensitivity information from multiple coils
    - c. Compressed sensing: Using sparsity constraints for reconstruction
    - d. Deep learning: Using non-linear neural network trained with large datasets
  - (4) Hybrid techniques: Combining different acceleration strategies to achieve robust acceleration

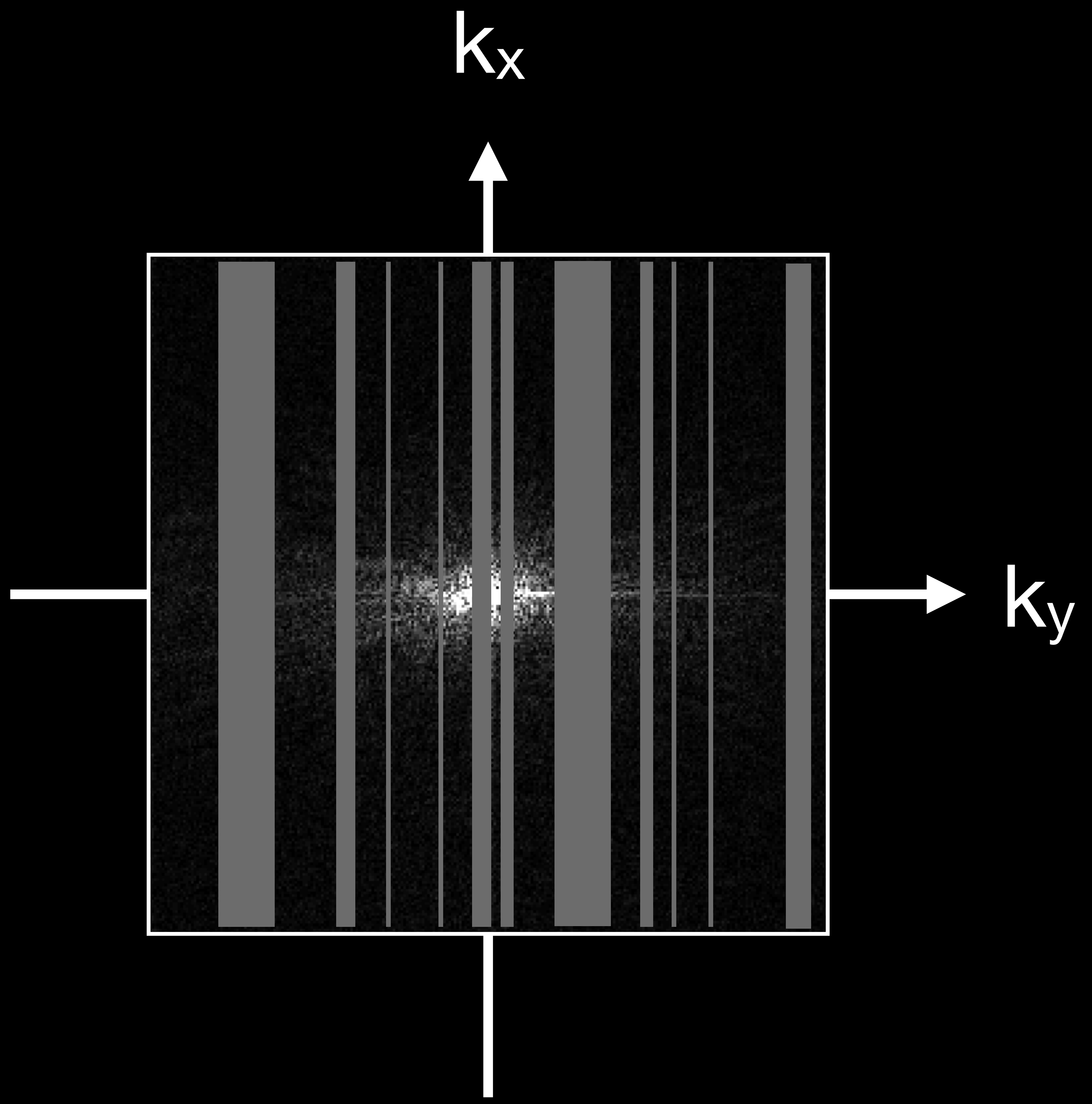
https://mriquestions.com/echo-planar-imaging.html

Underdetermined system

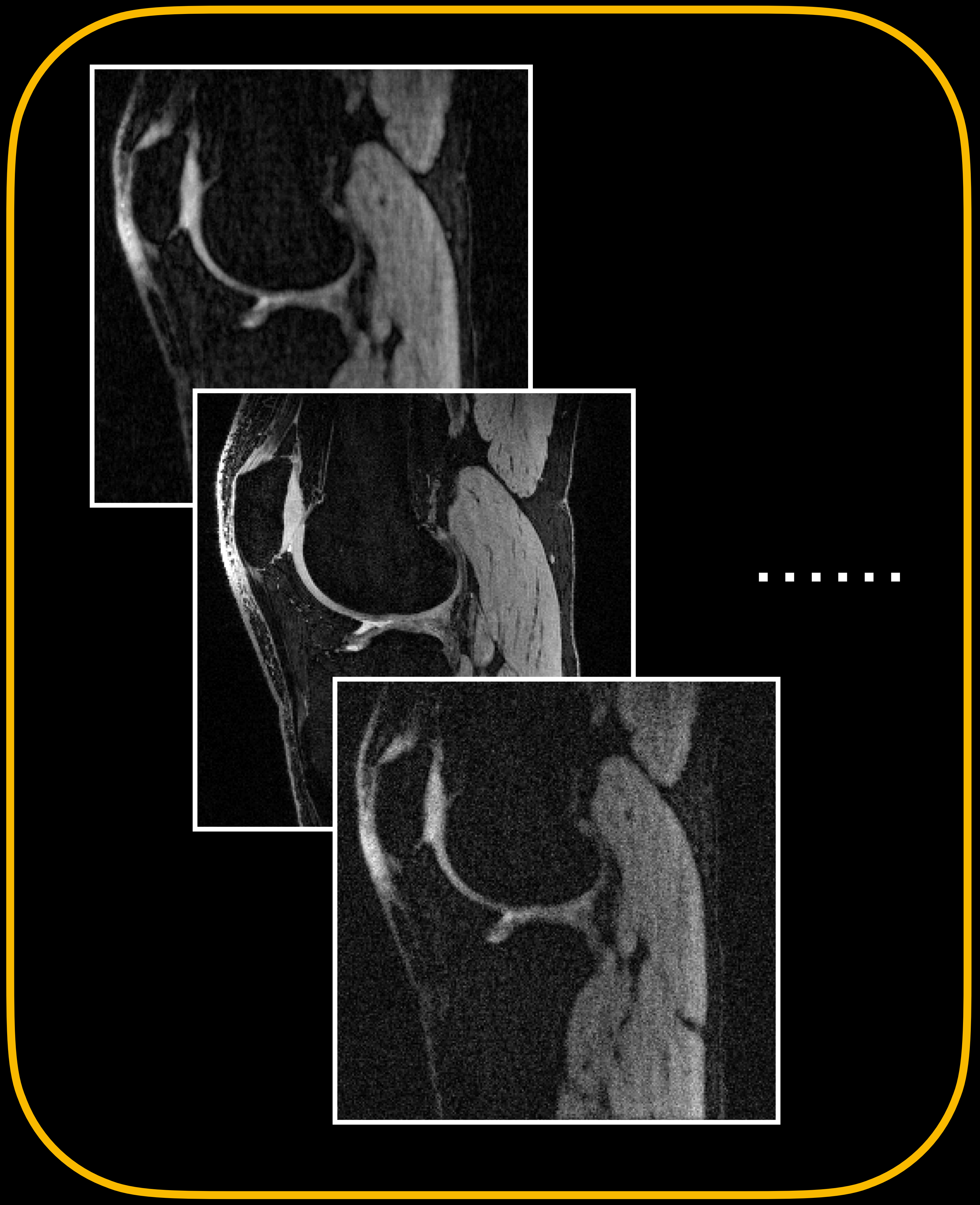
Fully-sampled
(Nyquist criteria fulfilled)



Undersampled
(Missing some k-space lines)



Images with the same undersampled k-space data



Use prior information about the images to help us solve the underdetermined problem

Compressed sensing MRI

- Compressed sensing MRI can reconstruct an image with high fidelity from undersampled k-space data given
  - (1) the image has transform sparsity (or a sparse representation in some transform domain)
  - (2) the k-space sampling pattern generates incoherent artifacts in the sparse transform domain
- Compressed sensing MRI usually involves a nonlinear reconstruction method to recover the image

Sparse representation

- Many images have a sparse representation in some transform domain
- Example 1: Discrete cosine transform (DCT)
  - JPEG uses DCT for image compression

$$X_k = \sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right] \qquad for\ k = 0, \ldots N-1$$

Original image

2D DCT coefficients

Compressed image (3.7-fold)
by preserving large DCT coefficients

See code example 01

Sparse representation

- Example 2: Wavelet transform
  - JPEG 2000 uses Wavelet transform for image compression

Original image

2D Wavelet coefficients

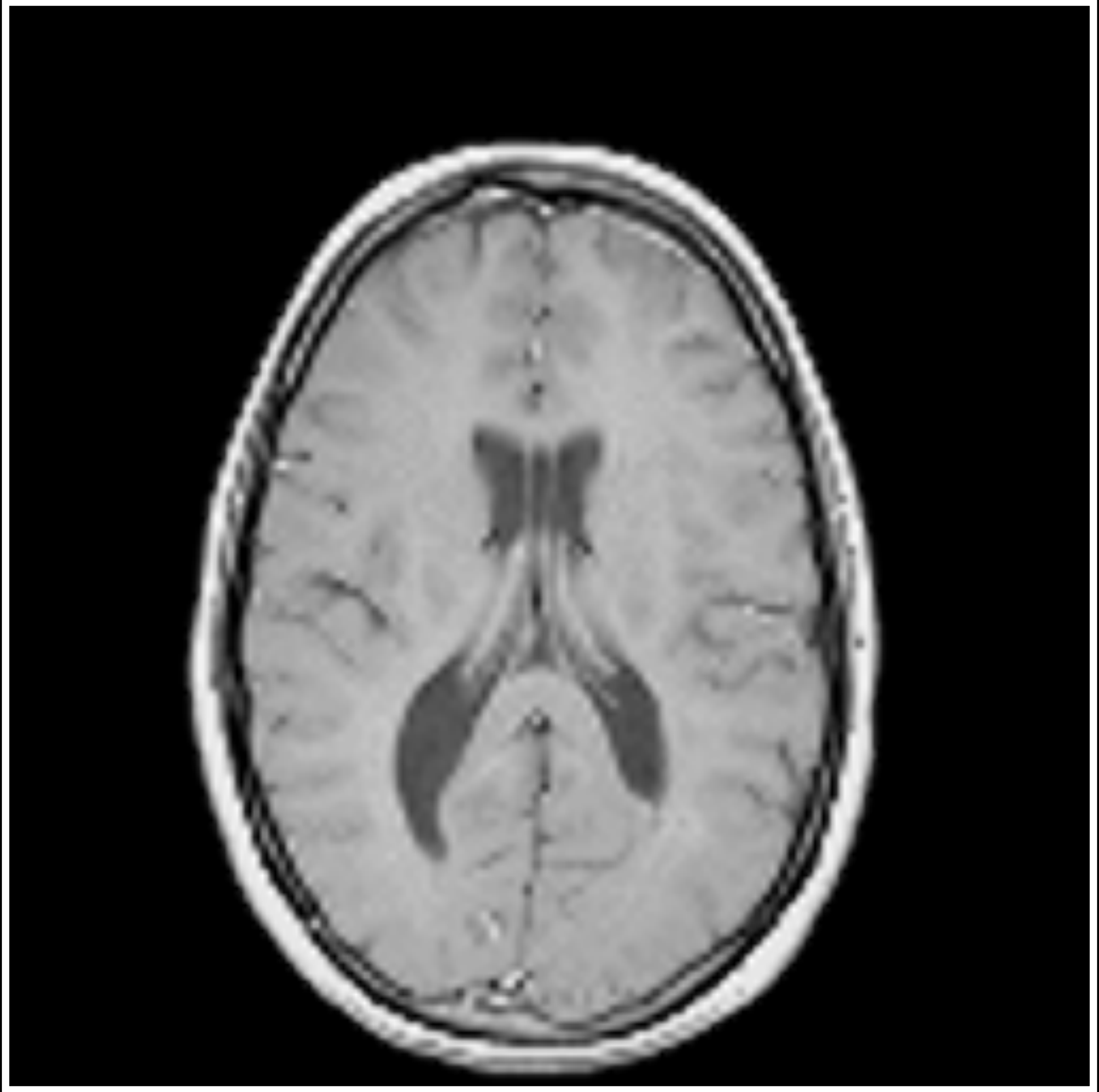
Compressed image (5.3-fold)
by preserving large Wavelet coefficients

See code example 02

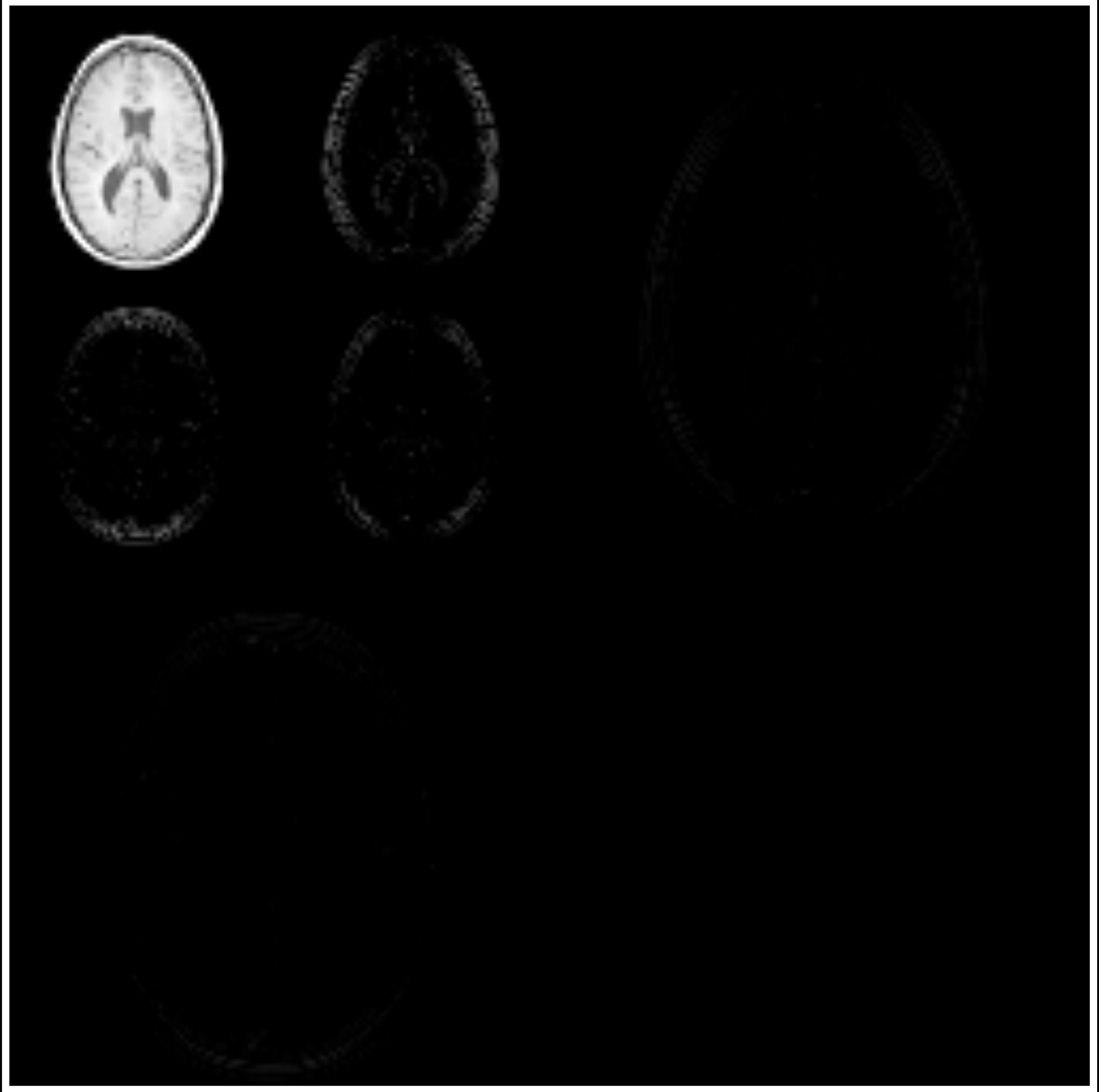
Sparse representation

- Example 3: Wavelet transform for a brain image

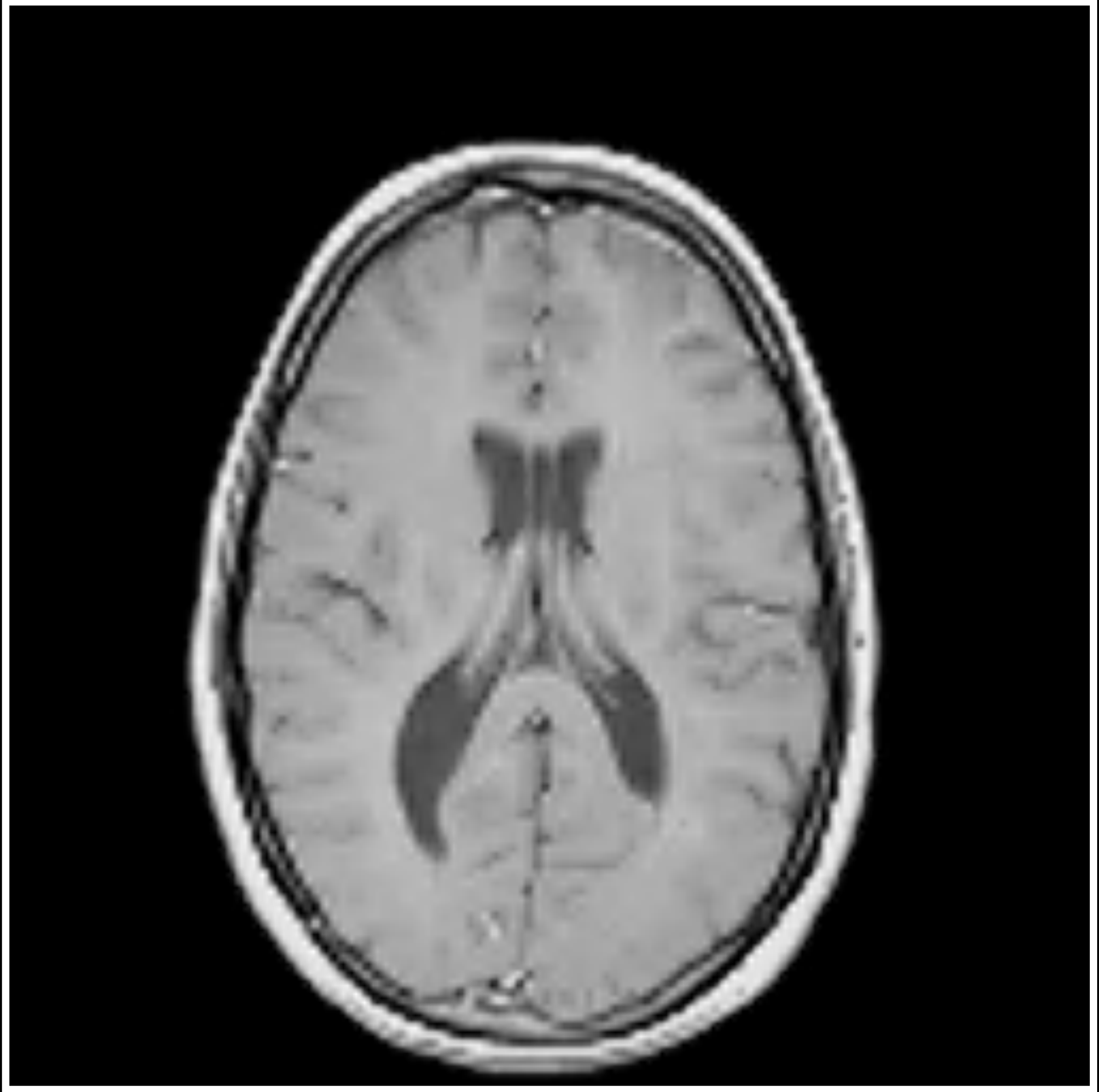
Original image



2D Wavelet coefficients



Compressed image (4.8-fold)
by preserving large Wavelet coefficients

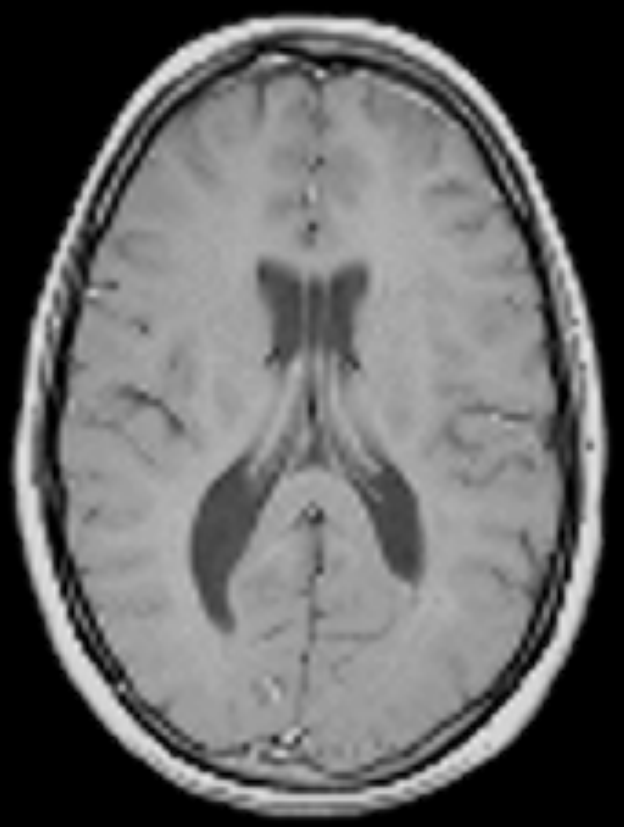


See code example 03

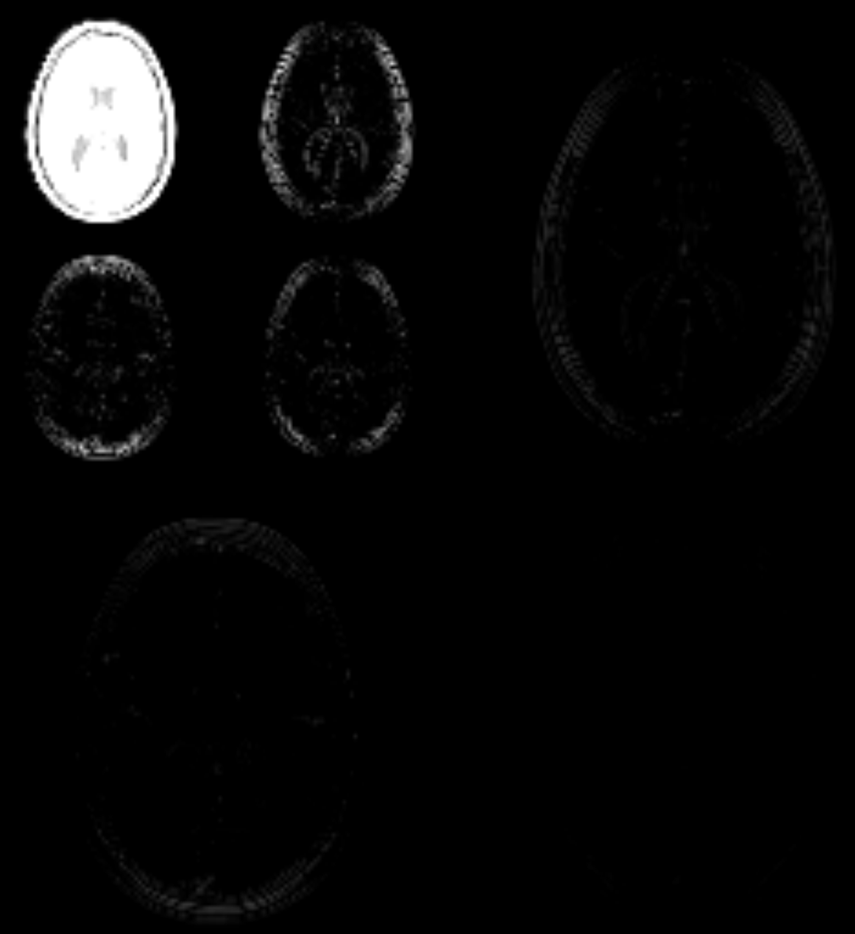
Sparse representation

- Many (MRI) images have a sparse representation in some transform domain

Brain image

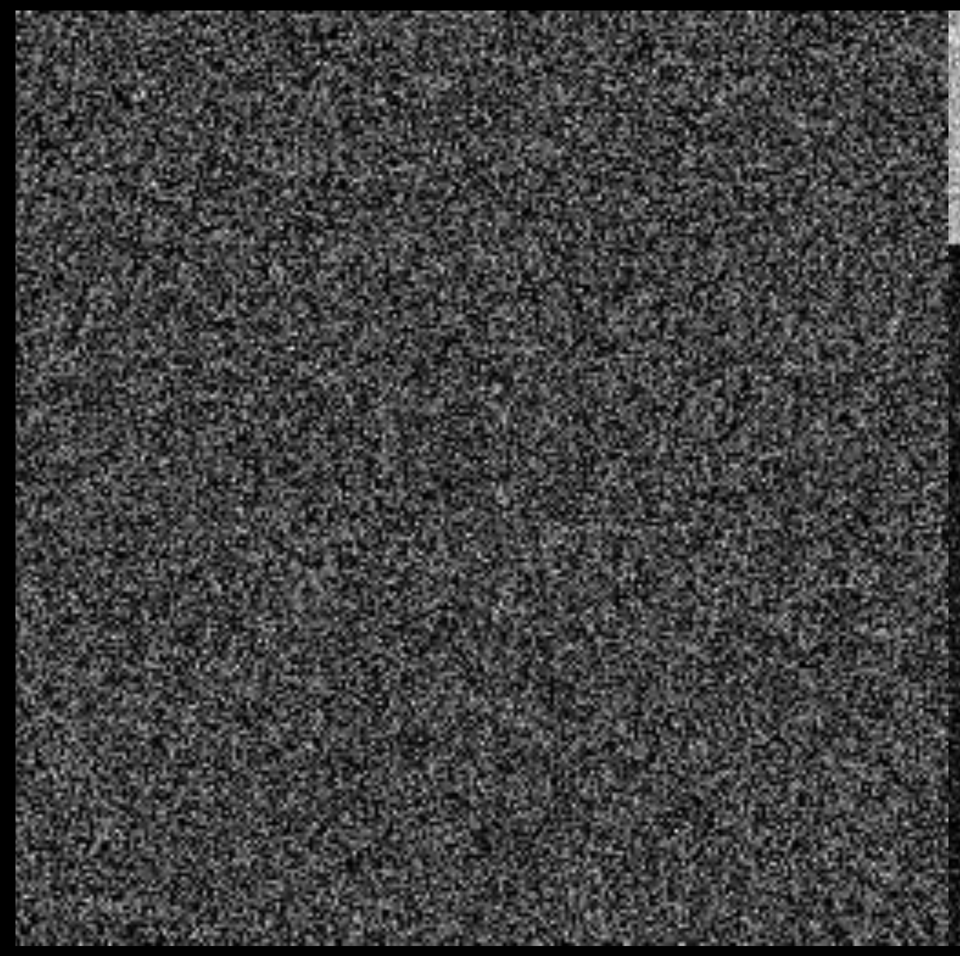


2D Wavelet coefficients
of a brain image

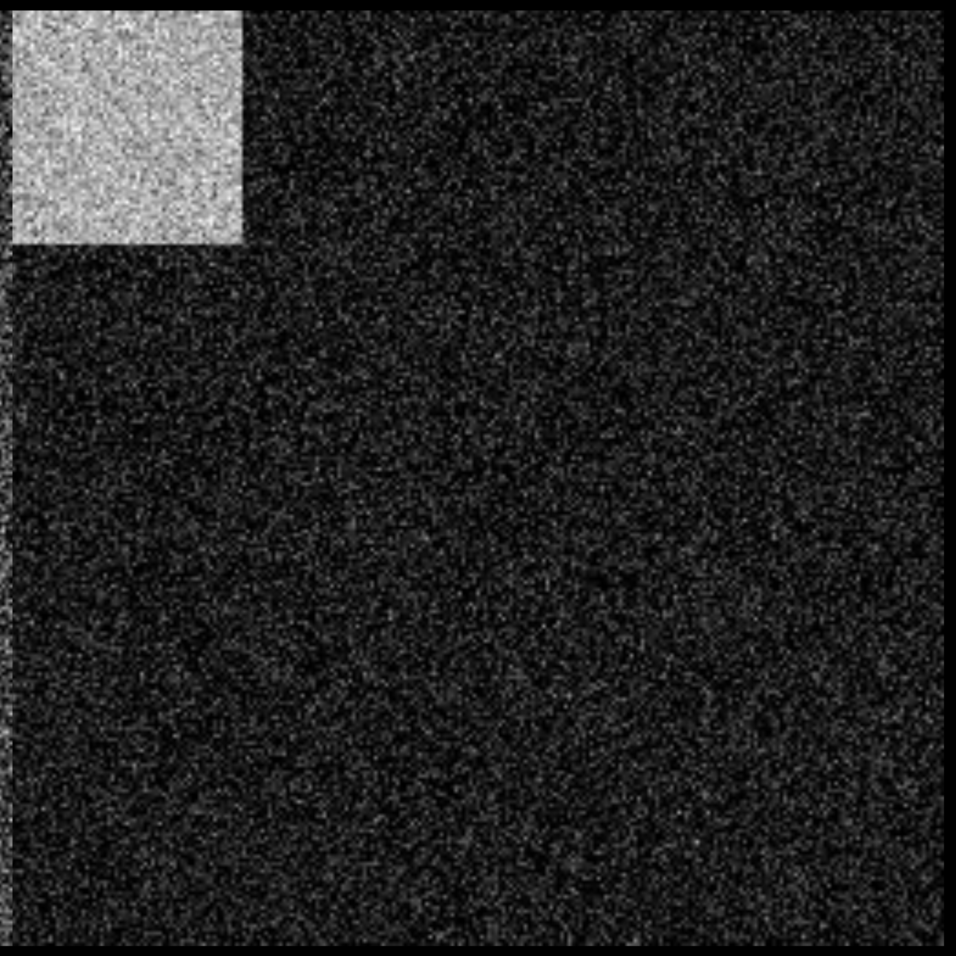


Sparse!

Noisy image



2D Wavelet coefficients
of a noisy image



Not so sparse…

See code example 04

Sparse representation

- How does this “prior information” can help in image reconstruction problem?
  - *When the reconstruction problem is under-determined, the corresponding artifact-free or fully sampled image that best matches the undersampled data will more likely be one that has a sparse representation*

See code example 04

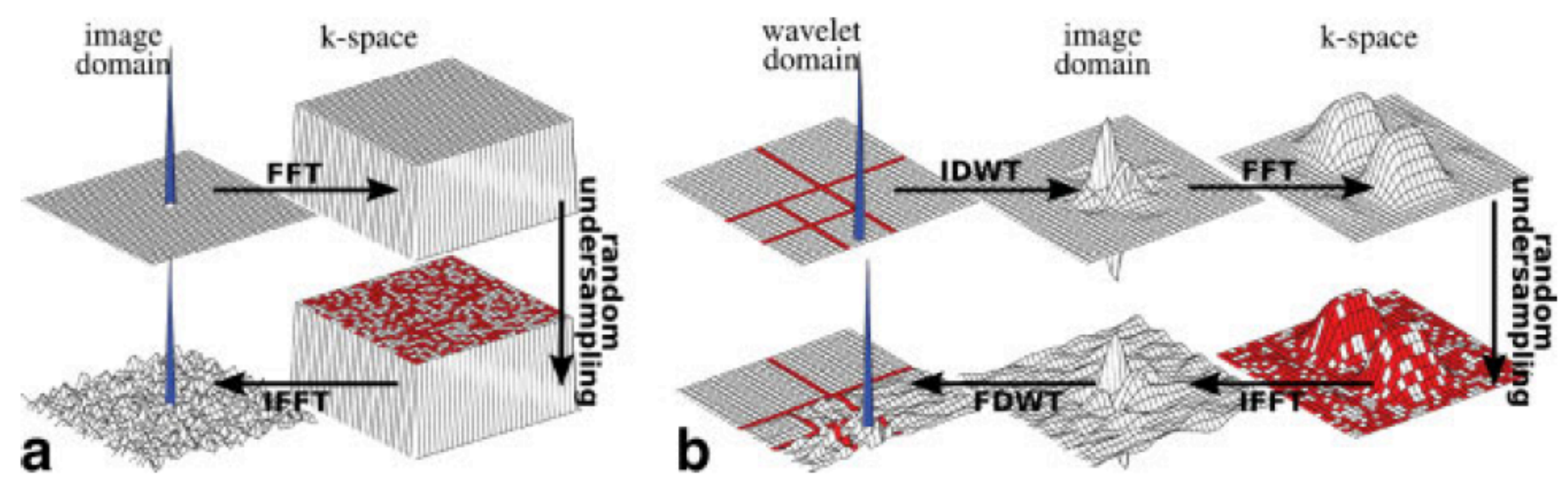
Incoherent artifacts

- The second requirement for compressed sensing MRI:
  - The undersampling pattern should generate incoherent artifacts in the sparse transform domain

- What are incoherent artifacts?
  - Noise-like or diffuse image artifacts that lack a clear, structured, or predictable pattern

Incoherent artifacts

- Using the point spread function to analyze



(Figure from: Lustig et al., MRM 2007)

Incoherent artifacts

Inverse
Fourier transform

Fourier transform

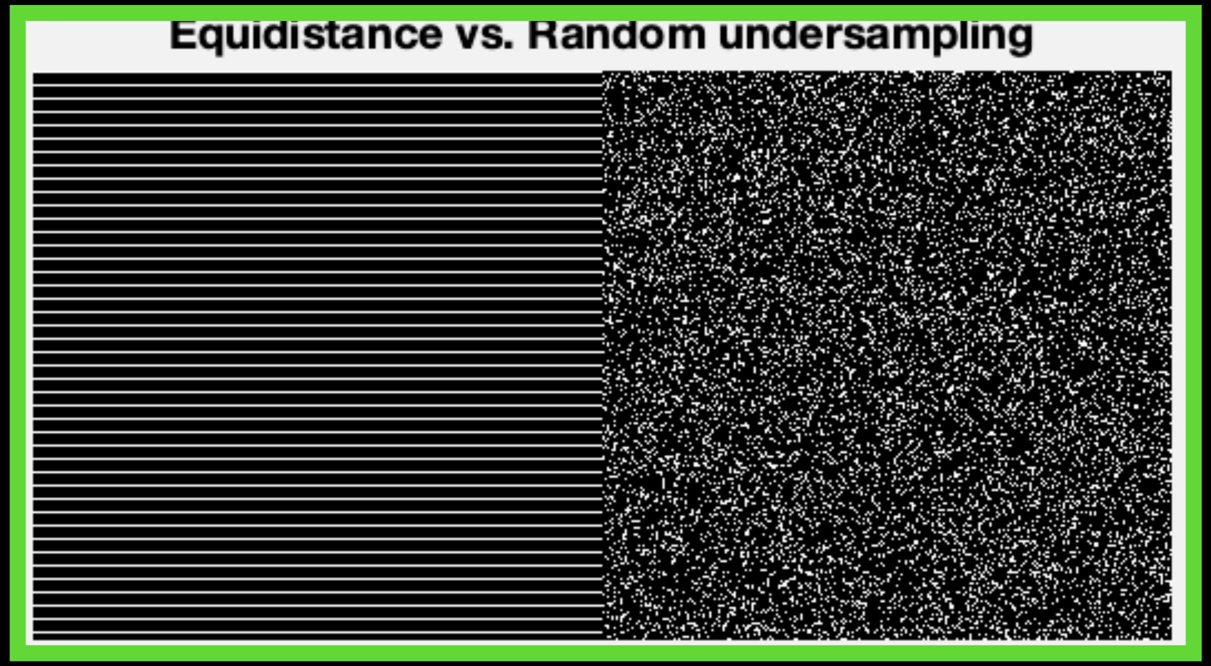
k-space domain

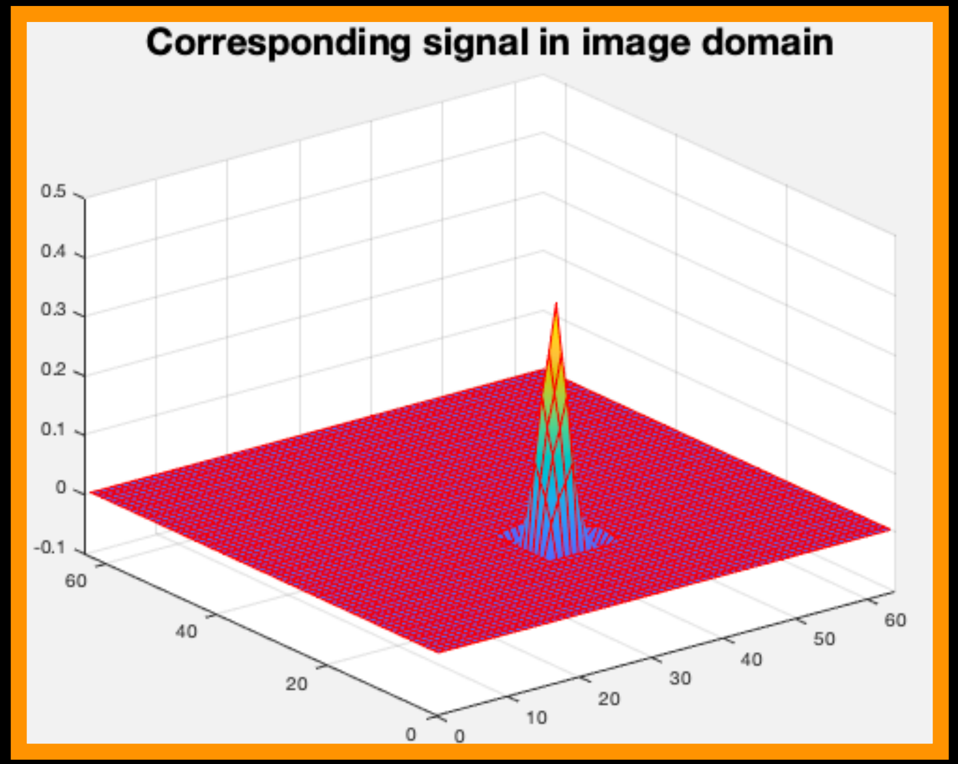
Image domain

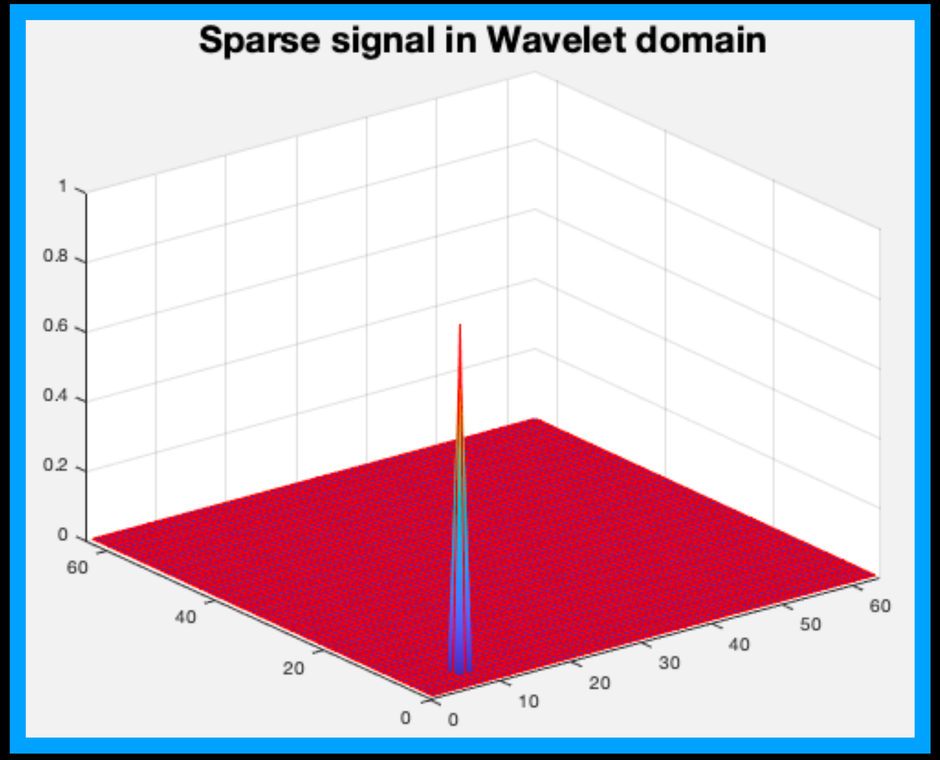
Wavelet domain

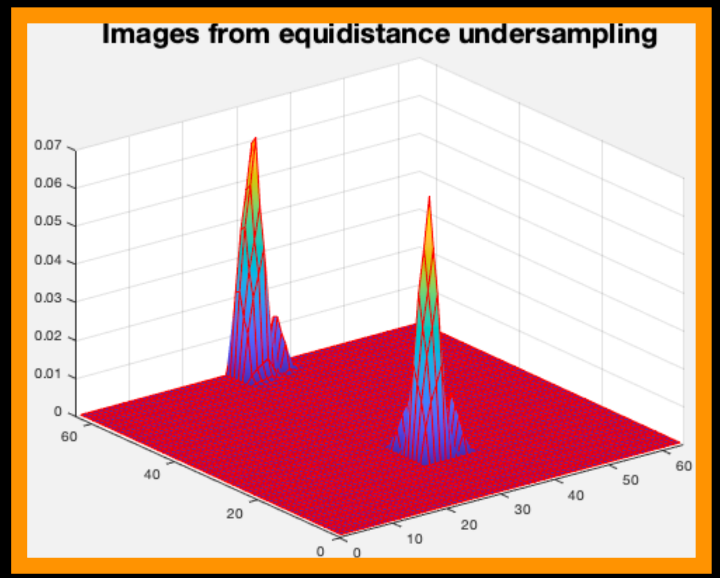
Wavelet transform

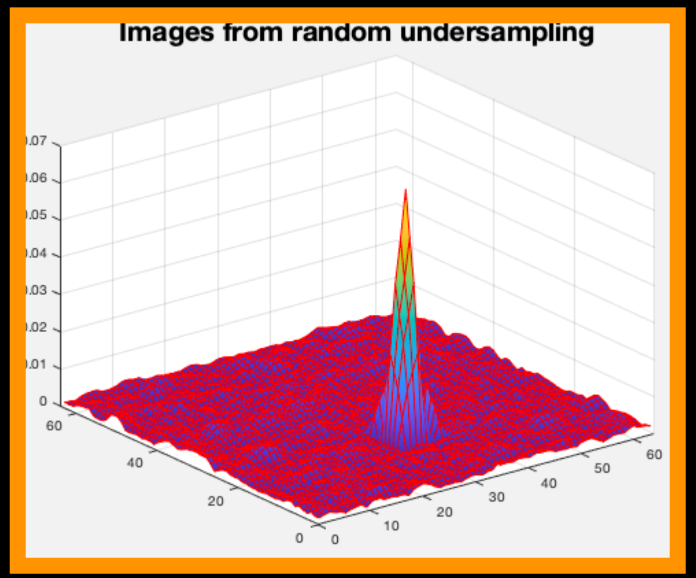
Inverse
Wavelet transform

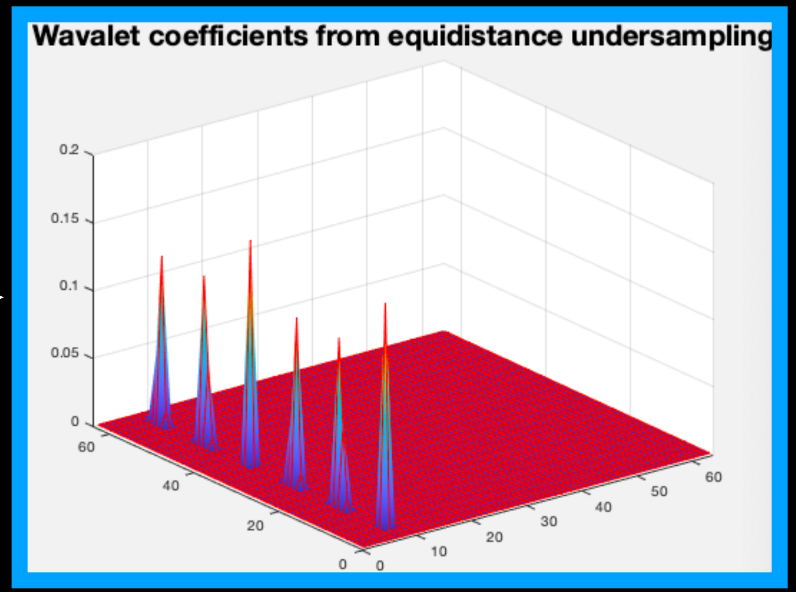


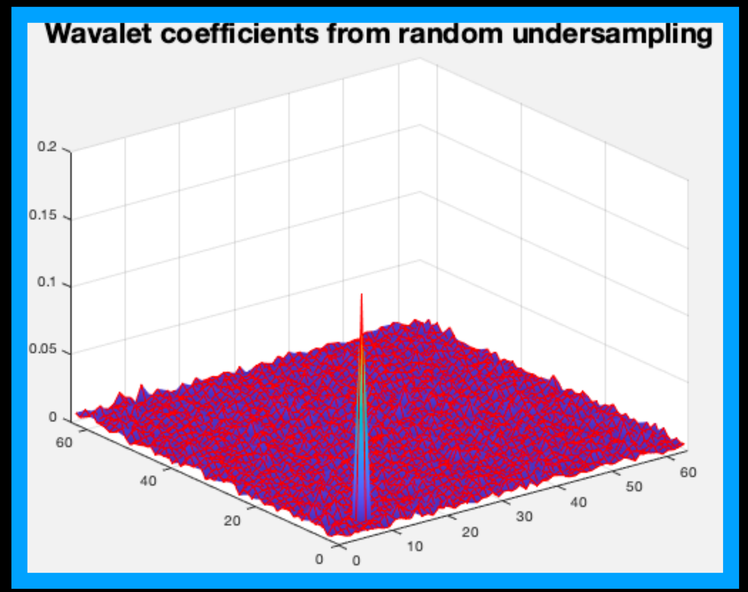












See code example 05

L0, L1 and L2 norm

- Vector norm: a method to measure the length of a vector
- L0 norm ($\|x\|_0$): number of non-zero entries
- L1 norm ($\|x\|_1$): sum of absolute values of the entries

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$$

- L2 norm ($\|x\|_2$): square root of sum of squared values of the entries

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2}$$

L0, L1 and L2 norm

$$v_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 2 \\ -3 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 2 \\ -4 \\ 2 \end{bmatrix}$$

- $\| v_1 \|_2 = \sqrt{38}$
- $\| v_1 \|_1 = 10$
- $\| v_1 \|_0 = 3$

- $\| v_2 \|_2 = \sqrt{38}$
- $\| v_2 \|_1 = 14$
- $\| v_2 \|_0 = 6$

- Two vectors with similar energy (L2 norm) can have different levels of sparsity (L1 norm)

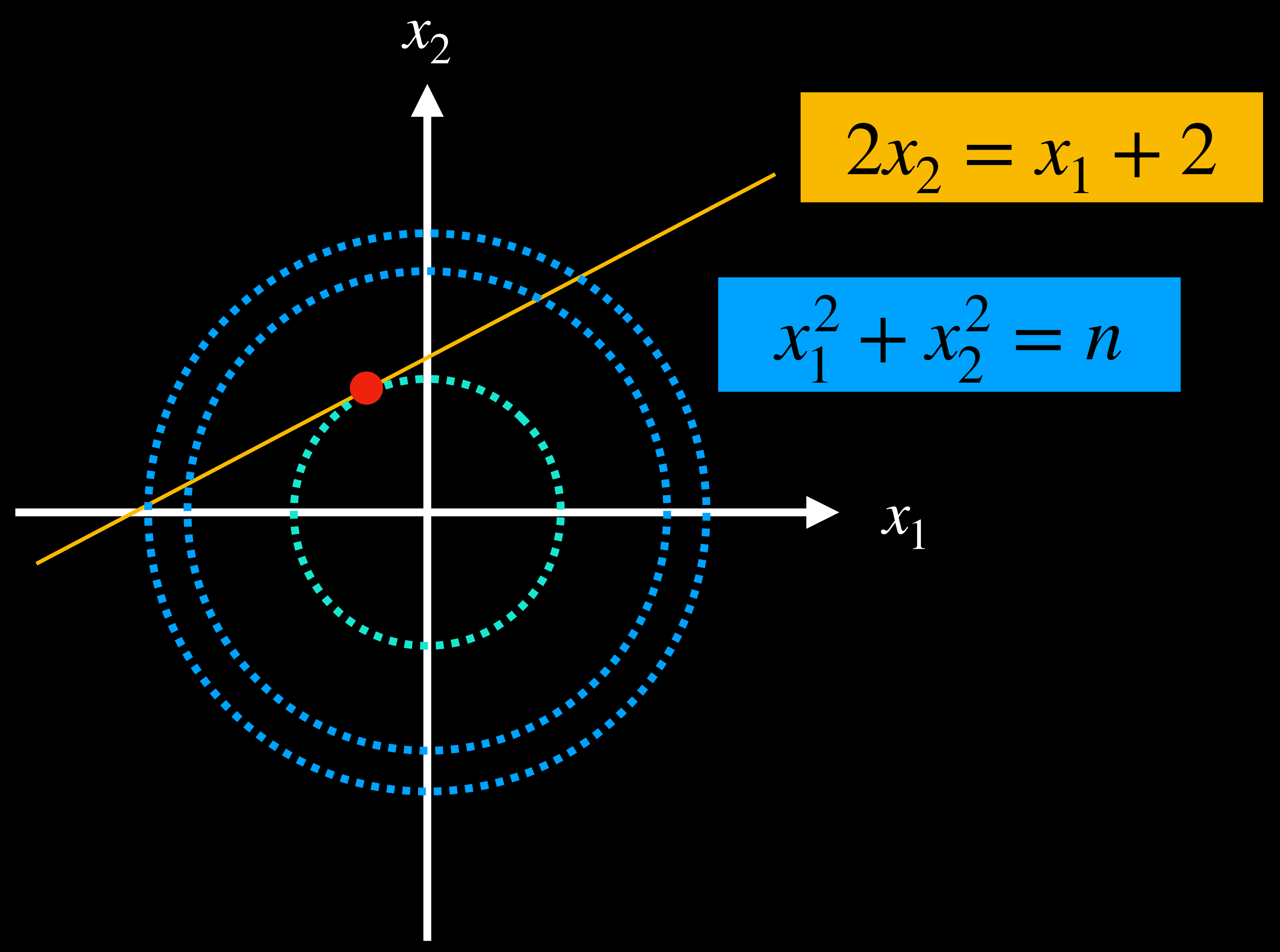
Exercises

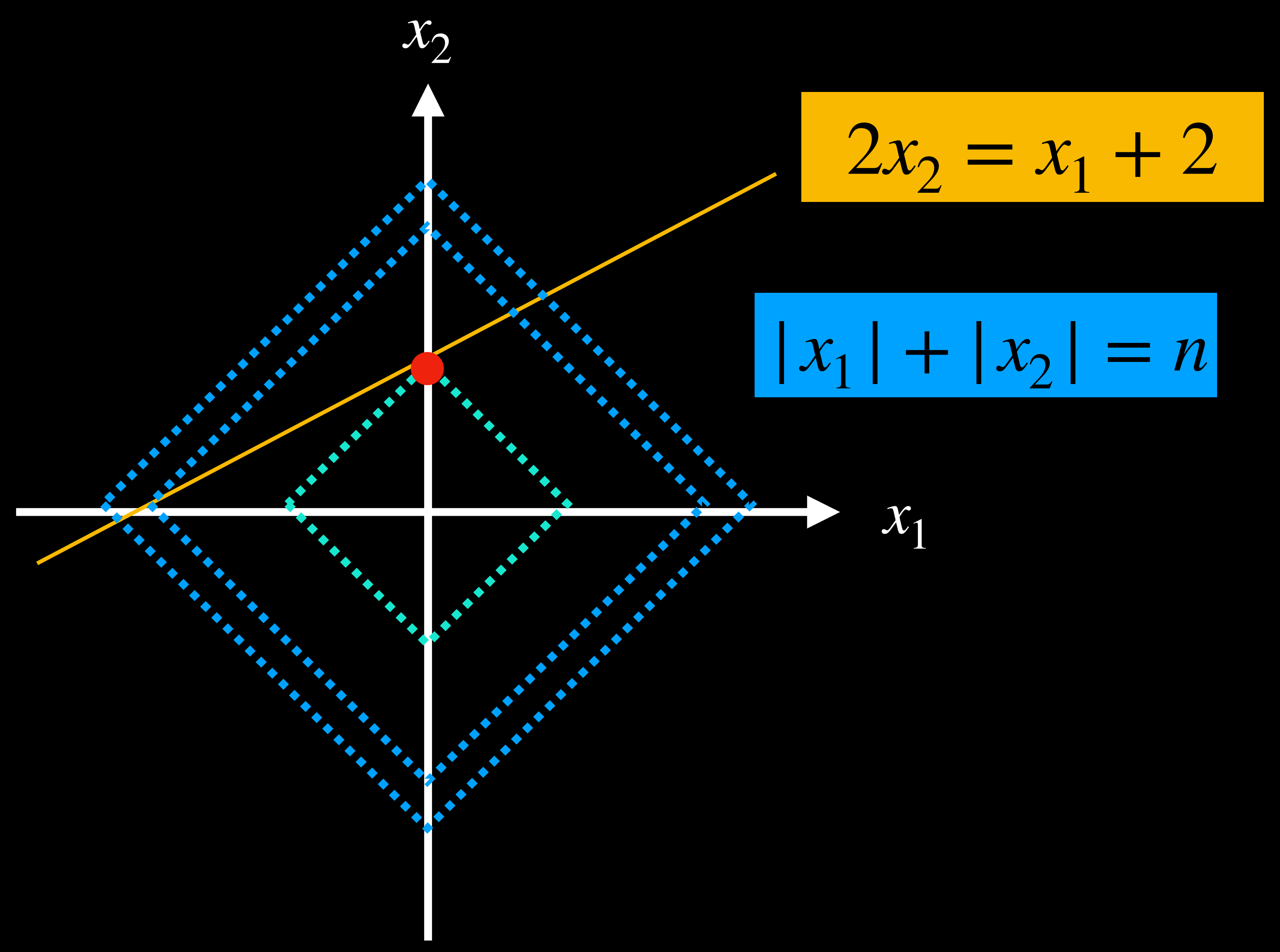
- Suppose we have a 2D vector $x = [x_1, x_2]$

- Exercise 1: $$\text{argmin}_x \quad ||x||_2 \quad s.t. \quad 2x_2 = x_1 + 2$$

- Exercise 2: $$\text{argmin}_x \quad ||x||_1 \quad s.t. \quad 2x_2 = x_1 + 2$$

- Example 3: $$\text{argmin}_x \quad ||x||_0 \quad s.t. \quad 2x_2 = x_1 + 2$$





Exercises

- L2 norm minimization: Find a solution with smallest energy
- L1 and L0 norm minimization: Find a sparse solution

Mathematical formulation

Our goal: Find an image that has the sparsest coefficients in the Wavelet domain and the image is consistent with the undersampled k-space data

Turn into an optimization problem →

$$\mathrm{argmin}_x \; \| Wx \|_0 \quad \text{subject to} \quad \| Fx - y \|_2 < \epsilon$$

Convex relaxation using L1 norm →

$$\mathrm{argmin}_x \; \| Wx \|_1 \quad \text{subject to} \quad \| Fx - y \|_2 < \epsilon$$

W: Wavelet transform operator
x: reconstructed image
F: Fourier transform operator
y: acquired undersampled k-space data

Mathematical formulation

W: Wavelet transform operator
x: reconstructed image
F: Fourier transform operator
y: acquired undersampled k-space data
λ: regularization parameter
U: k-space sampling pattern

$$argmin_x \quad \| Wx \|_1$$
$$subject\ \ to \quad \| Fx - y \|_2 < \epsilon$$

Use Lagrangian form

$$\rightarrow \quad argmin_x \quad \| Fx - y \|_2^2 + \lambda \| Wx \|_1$$

Explicitly include an sampling operator

$$\rightarrow \quad argmin_x \quad \| UFx - y \|_2^2 + \lambda \| Wx \|_1$$

Mathematical formulation

W: Wavelet transform operator
x: reconstructed image
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λ: regularization parameter
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Use Lagrangian form

$$\rightarrow \quad argmin_x \quad \| Fx - y \|_2^2 + \lambda \| Wx \|_1$$

Explicitly include an sampling operator

$$\rightarrow \quad argmin_x \quad \underline{\| UFx - y \|_2^2 + \lambda \| Wx \|_1}$$

Cost function

Optimization algorithm

- Solving $\min \| UFx - y \|_2^2 + \lambda \| Wx \|_1$ is non-trivial since the cost function is not smoothed at Wx=0

- Different approaches have been used to solve $\min \| UFx - y \|_2^2 + \lambda \| Wx \|_1$
  - Conjugate gradient descent[1]
  - ADMM[2,3]
  - Primal-dual algorithm[4]
  - …

[1] Lustig et al., Magn Reson Med. 2007;58(6):1182-95
[2] Wang et al., SIAM J Imag Sci. 2008;1(3):248-72
[3] Ramani et al., IEEE Trans Med Imaging. 2011;30(3):694-706
[4] Chambolle et al., J Math Imaging Vision. 2011;40(1):120-45

Optimization algorithm

- Conjugate gradient descent

$$argmin_m \quad f(m) = \| UFm - y \|_2^2 + \lambda \| Wx \|_1$$

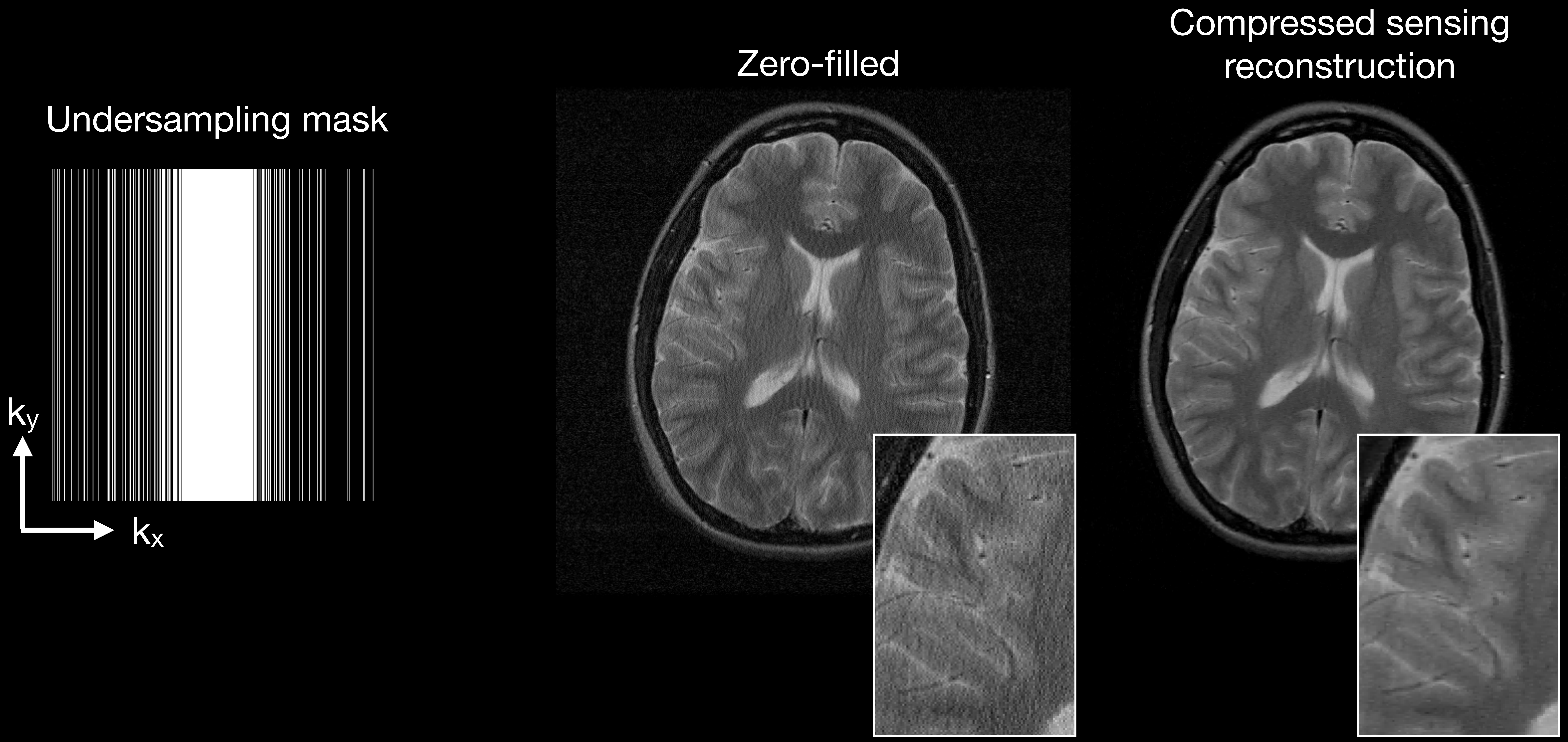
% Initialization
$k = 0;\ m = 0;\ g_0 = \nabla f(m_0);\ \Delta m_0 = -g_0$
% Iterations
while $(||g_k||_2 < \text{TolGrad and } k > \text{maxIter})$ {
  % Backtracking line-search
  $t = 1;$ *while* $(f(m_k + t\Delta m_k) > f(m_k) + \alpha t \cdot Real(g_k^* \Delta m_k))$
    $\{t = \beta t\}$
  $m_{k+1} = m_k + t\Delta m_k$
  $g_{k+1} = \nabla f(m_{k+1})$
  $\gamma = \frac{||g_{k+1}||_2^2}{||g_k||_2^2}$
  $\Delta m_{k+1} = -g_{k+1} + \gamma \Delta m_k$
  $k = k + 1$ }

g_k: gradient at k[th] iteration
m_k: updated image result at k[th] iteration
TolGrad: stopping criteria
MaxIter: stopping criteria on iterations
α, β: line search parameters

From: Lustig et al., MRM 2007

Compressed sensing MRI

- Let’s run code to reconstruct images using compressed sensing…
  (see *code example 06*)



Compressed sensing MRI

- Compressed sensing MRI can reconstruct an image with high fidelity from undersampled k-space data given
  - (1) the image has transform sparsity (or a sparse representation in some transform domain)
  - (2) the k-space sampling pattern generates incoherent artifacts in the sparse transform domain
- Compressed sensing MRI usually involves a nonlinear reconstruction method to recover the image

Choice of regularization parameters

$$argmin_x \quad \| UFx - y \|_2^2 + \lambda \| Wx \|_1$$

- Many compressed sensing methods require tuning of regularization parameters.
- Larger weights on the sparsity term (larger λ):
  - Better suppression on noise or artifacts / Improved perceived SNR
  - Features more likely to be over-smoothed / Resulting in images with artificial appearance
- The regularization parameter is dataset-dependent.
- Methods for automatic regularization parameters selection have been investigated.

Compressed sensing + Parallel imaging

- Parallel imaging: Use information from multiple coils (e.g., coil sensitivity in SENSE reconstruction)
- Compressed sensing: Use sparsity constraints

- Combination of these two techniques:

Coil sensitivity maps

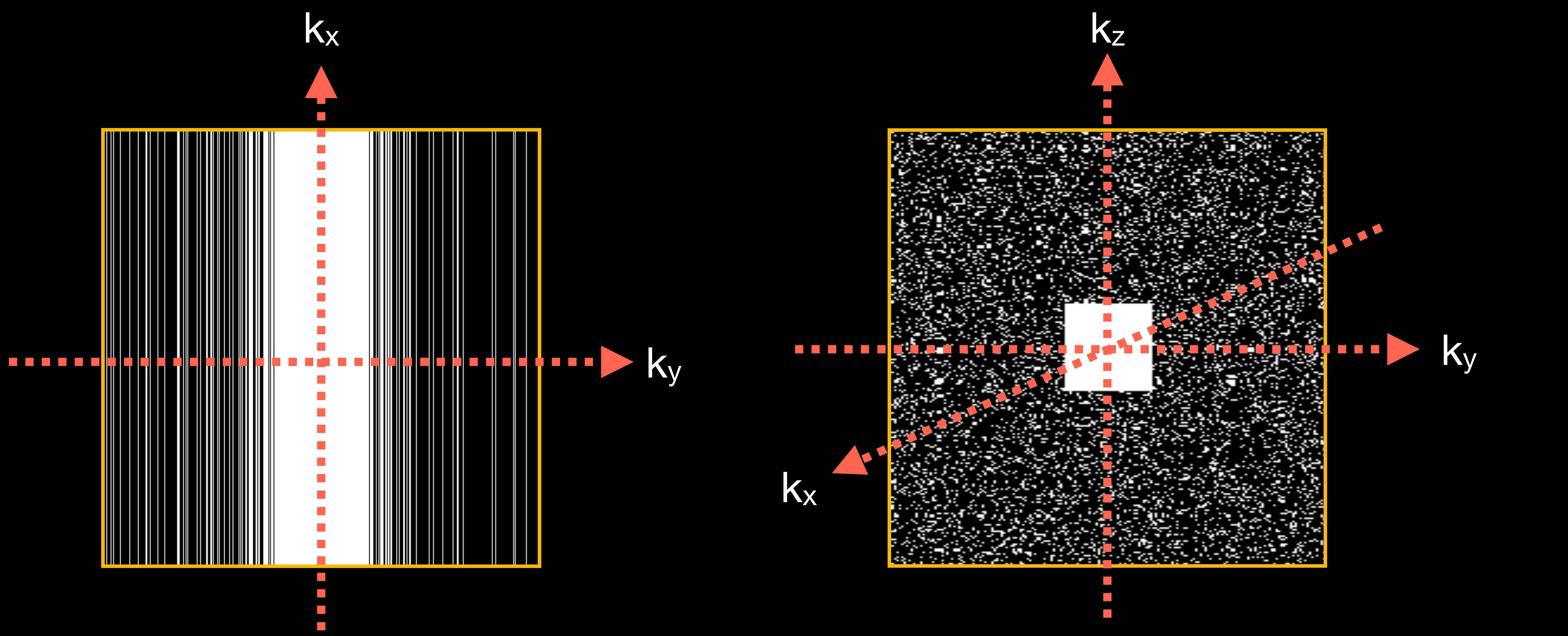
$$argmin_x \quad \| UFSx - y \|_2^2 + \lambda \| Wx \|_1$$

Coil combined image

Multi-coil k-space data

Compressed sensing + Parallel imaging

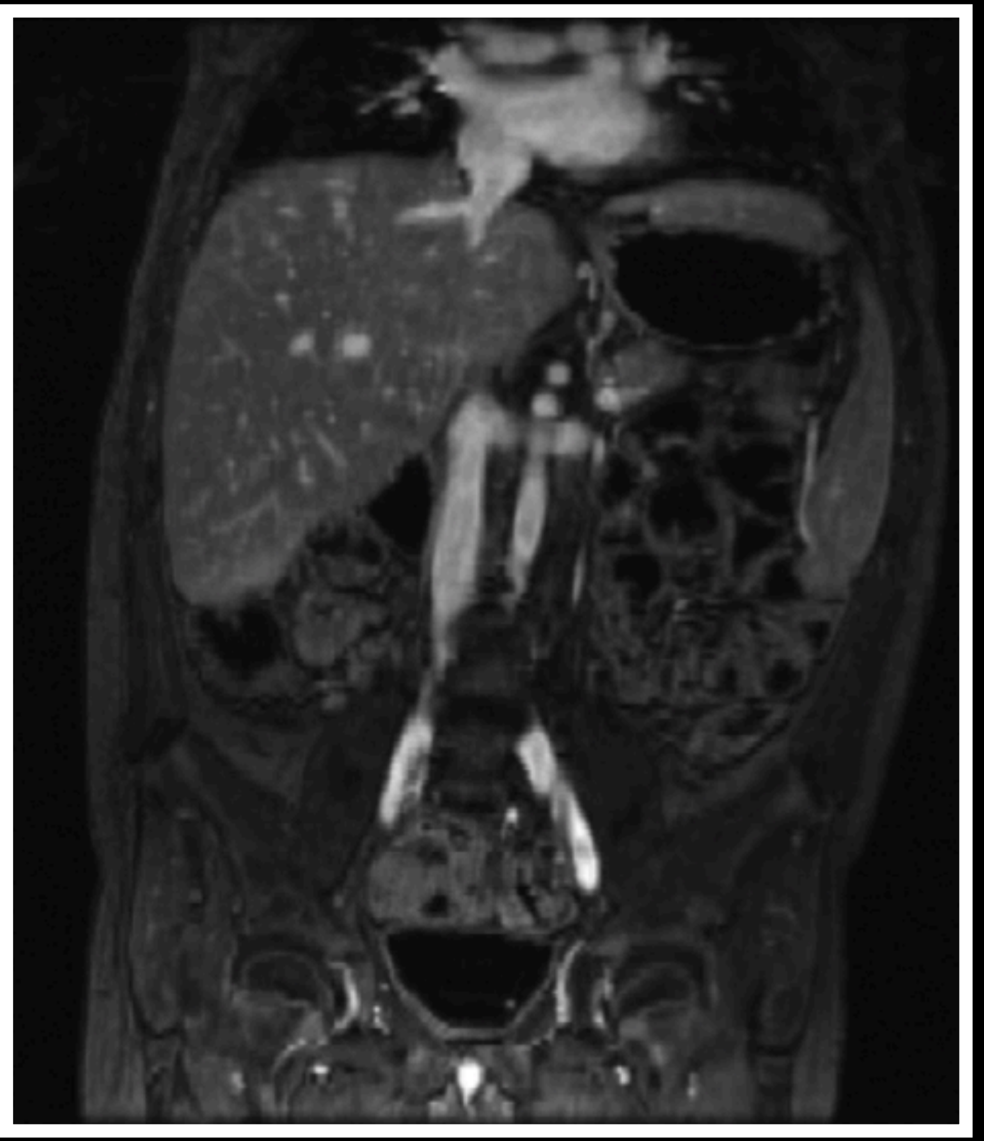
- Sampling trajectory:
  - The fully sampled region can be used to estimate coil sensitivity maps
  - The overall sampling scheme needs to generate incoherent under sampling artifacts



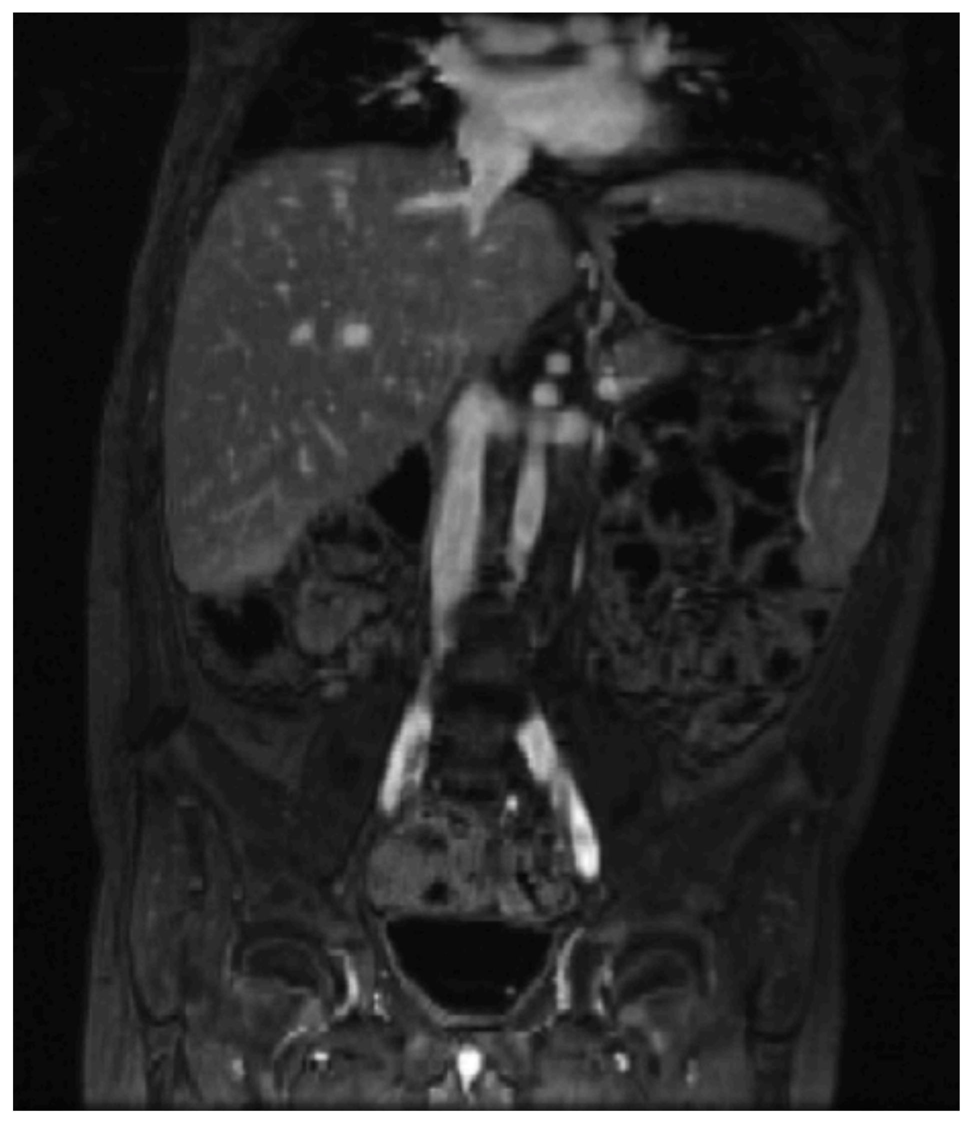
Coil compression

- A problem in applying compressed sensing reconstruction in some applications is the increased memory requirement and computational complexity due to a large number of coils.
- Coil compression (e.g., singular value decomposition-based technique) can be used to reduce the number of coils before compressed sensing reconstruction.

Reference (32 coil elements)



Coil-compressed image (6 virtual coils)



Error 20x

(Figures from: Zhang et al., MRM 2013)

Example (1): Knee T_2 mapping

- T_2 values in the knee cartilage have been used to detect disease and treatment changes in articular cartilage.
- T_2 quantification in the knee cartilage can help depict early cartilage degeneration.
- Challenges: Conventional multi-echo spin echo-based sequences are slow

Multi-echo spin-echo images

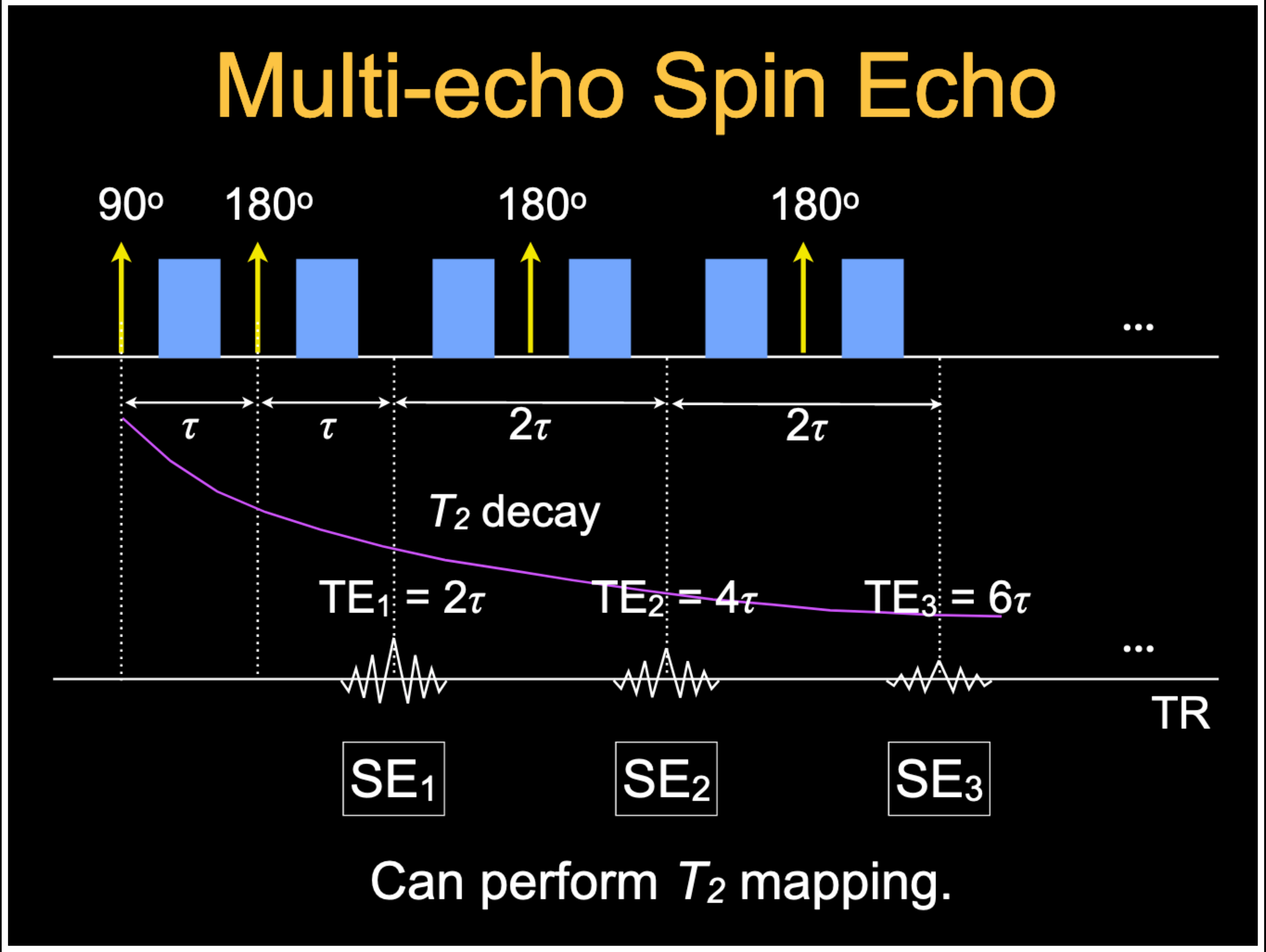
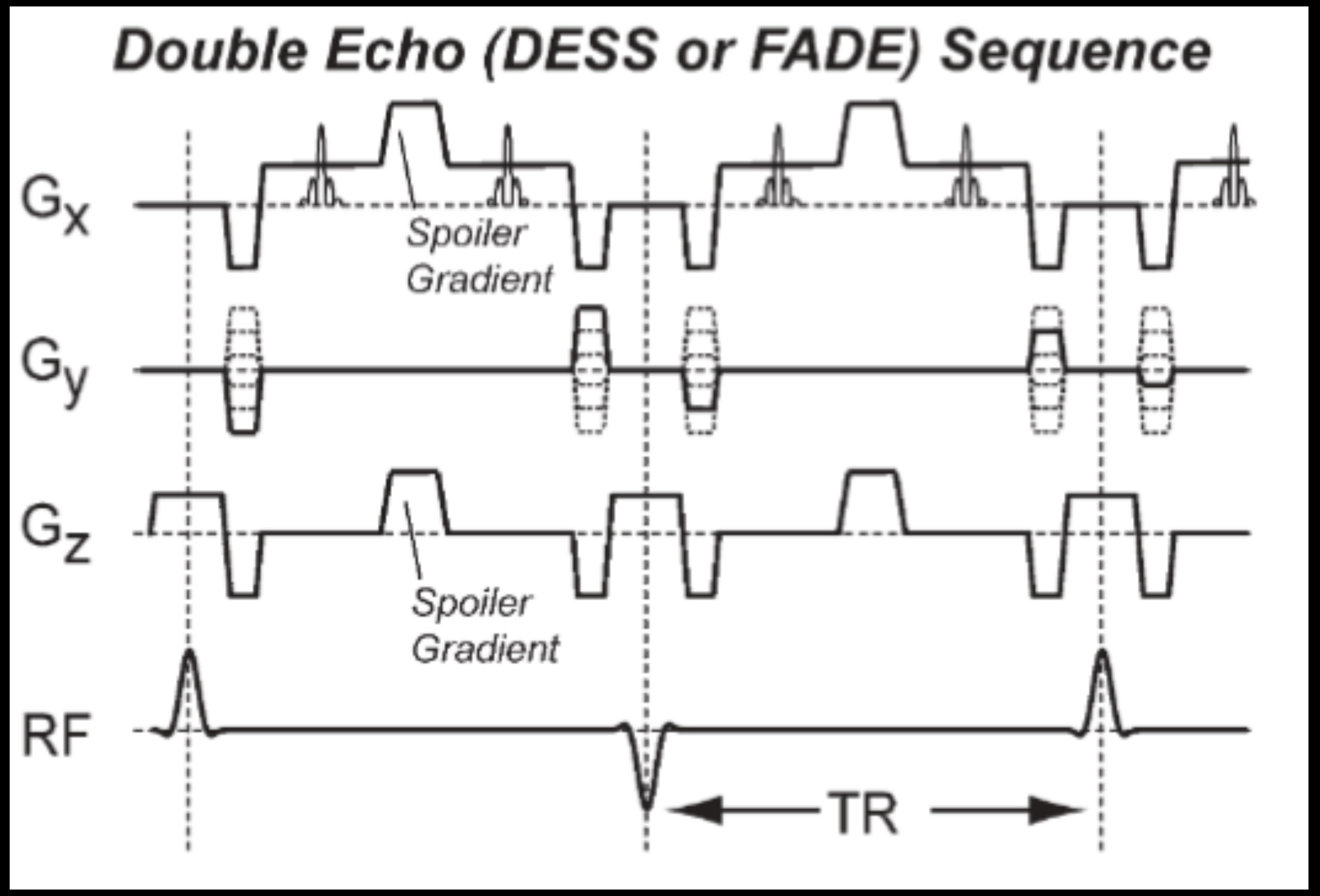


Figure from previous lecture slide

Example (1): Knee T_2 mapping

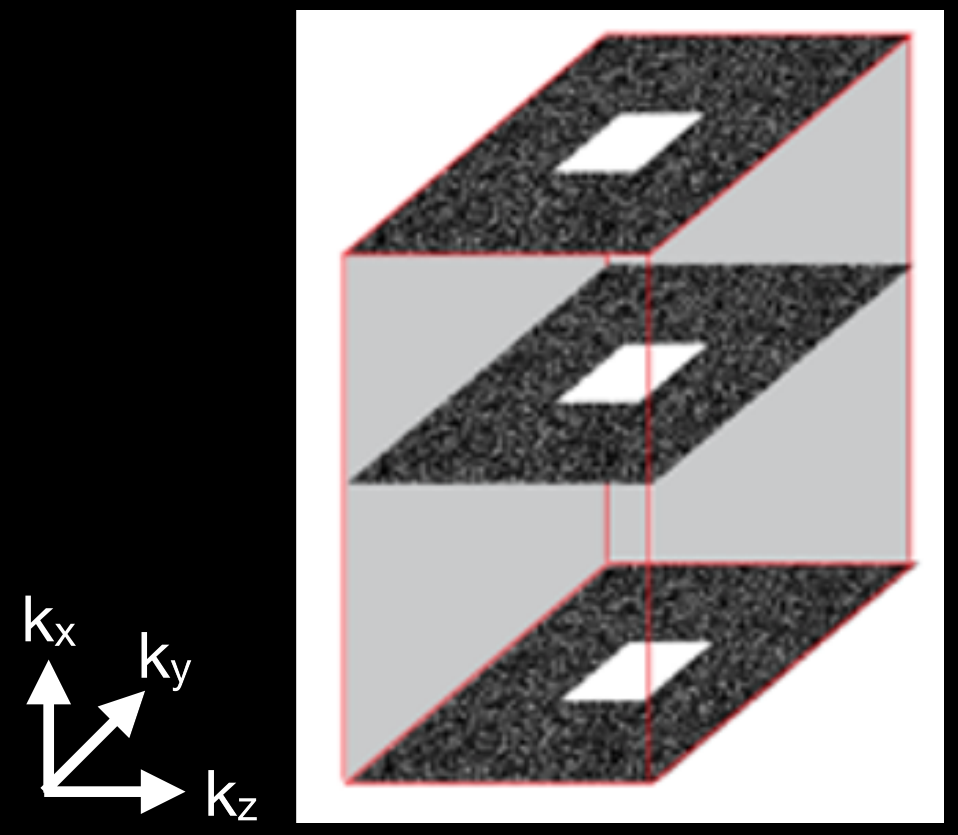
- Acceleration strategies
  - (1) Use a faster sequence: DESS (double/dual echo steady state)
  - (2) Use compressed sensing to accelerate

An extension to the gradient-spoiled GRE which acquires both SSFP-FID and SSFP-Echo



The difference between the two contrasts can be used to quantify T_2
(figure from Hargreaves et al., JMRI 2012)

Variable density sampling

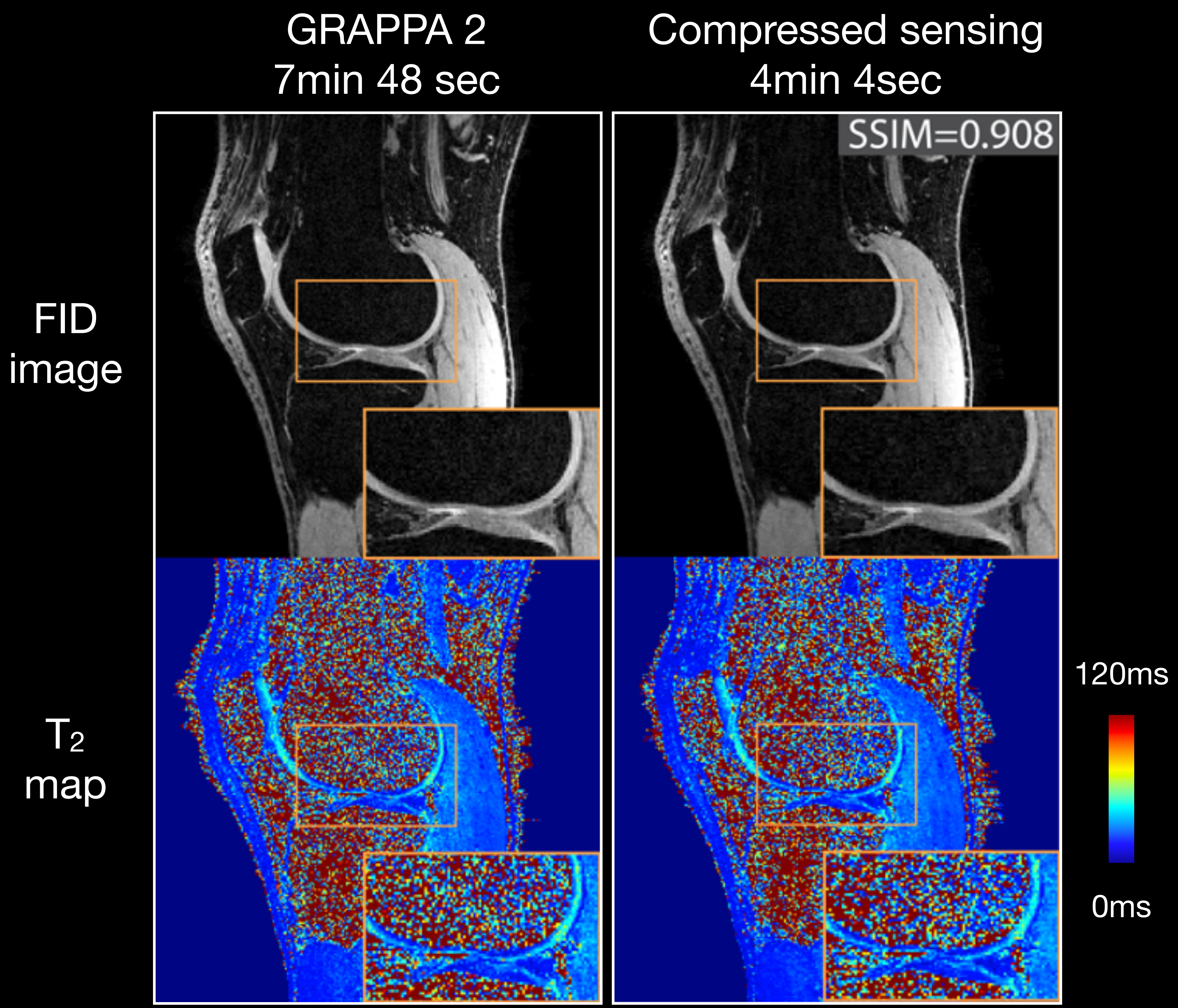


U: k-space sampling pattern
F: Fourier transform operator
S: coil sensitivity maps
x: reconstructed image
y: acquired undersampled k-space data
W: Wavelet transform operator
D: total variation operator
λ_1, λ_2: regularization parameters

Cost function

$$argmin_x \| UFSx - y \|_2^2 + \lambda_1 (\| Wx_{fid} \|_1 + r \| Wx_{echo} \|_1) + \lambda_2 (\| Dx_{fid} \|_1 + r \| Dx_{echo} \|_1)$$

Example (1): Knee T_2 mapping



(Figures from: Shih et al., ISMRM 2023)

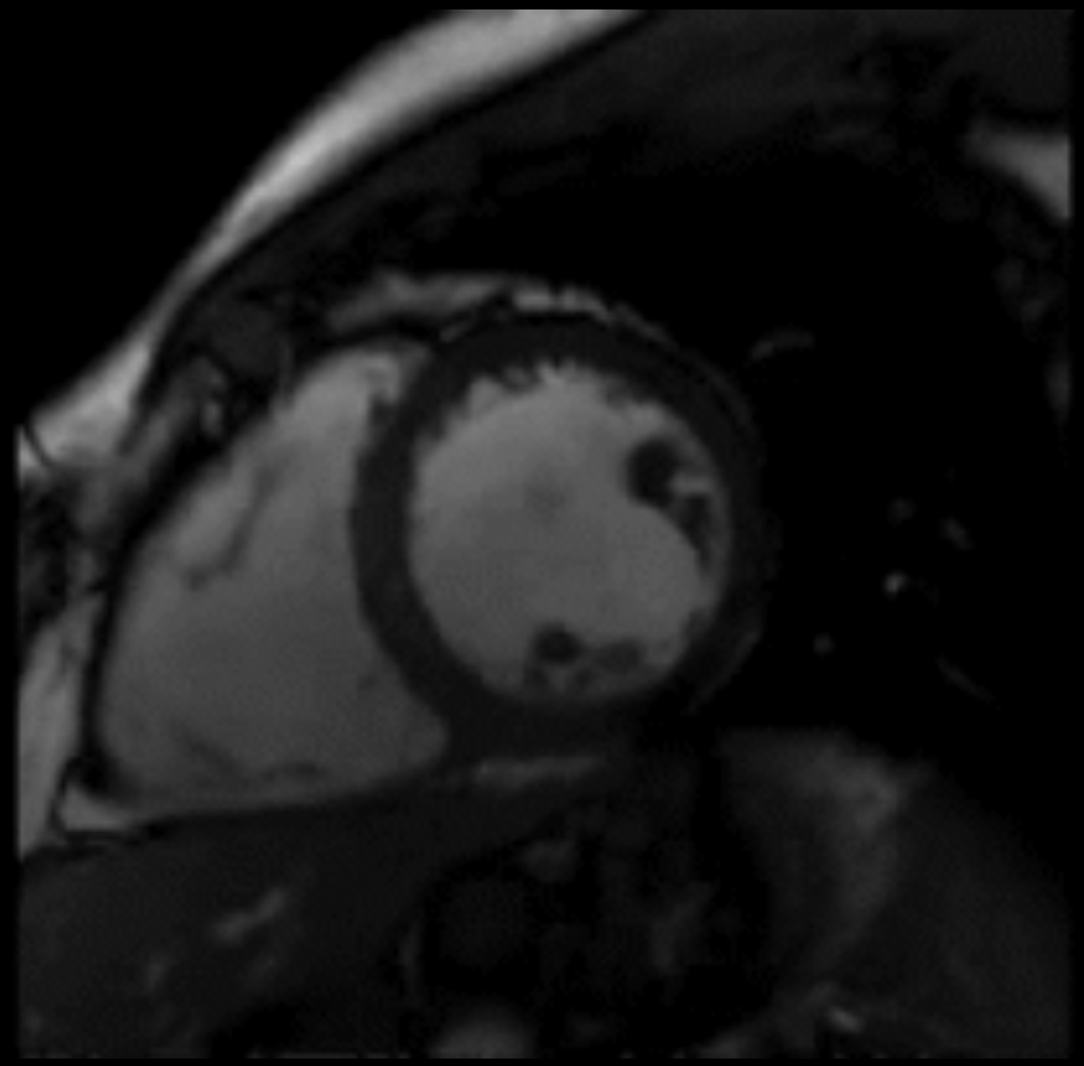
Example (1): Knee T_2 mapping

- Rapid knee cartilage T_2 mapping
  - *Constraint*: Sparsity in Wavelet transform and sparsity in total variation
  - *Data sampling*: variable density random sampling
  - *Optimization problem*: $argmin_x \| UFSx - y \|_2^2 + \lambda_1(\| Wx_{fid} \|_1 + r \| Wx_{echo} \|_1) + \lambda_2(\| Dx_{fid} \|_1 + r \| Dx_{echo} \|_1)$
  - *Reconstruction*: non-linear conjugate gradient method

U: k-space sampling pattern
F: Fourier transform operator
S: coil sensitivity maps
x: reconstructed image
y: acquired undersampled k-space data
W: Wavelet transform operator
D: total variation operator
λ_1, λ_2: regularization parameters

Example (2): Cardiac cine imaging

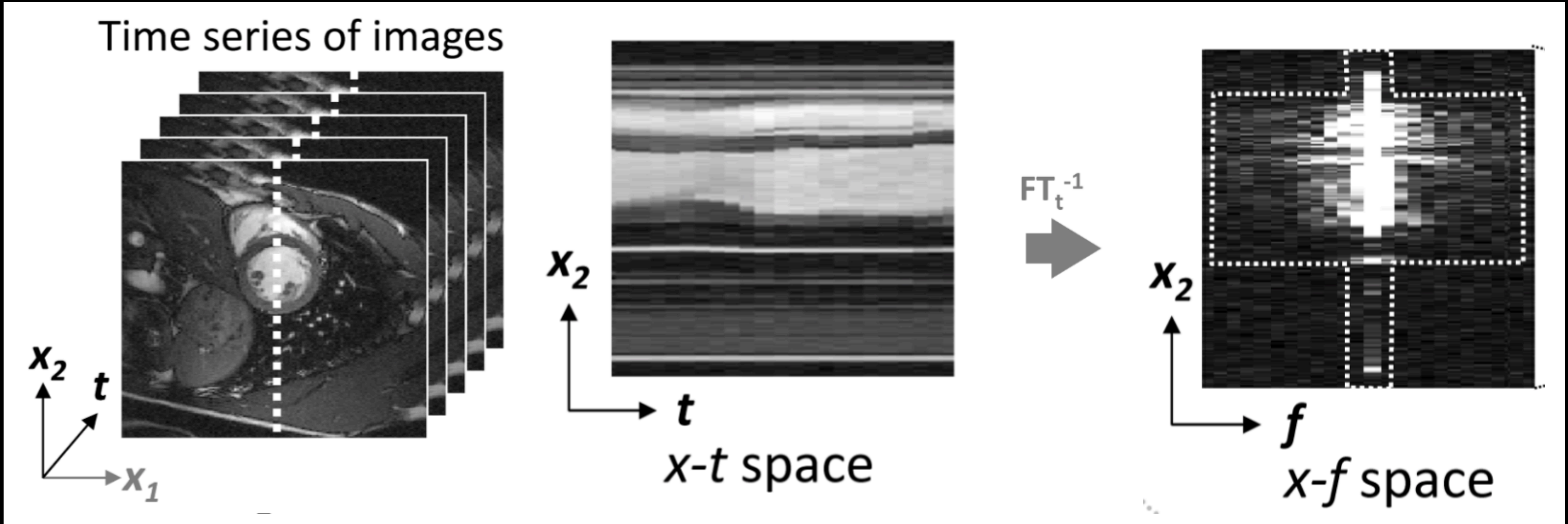
- Cardiac cine imaging for information of the heart function throughout the cardiac cycle
- Challenges: accelerating data acquisition without compromising the high resolution and image quality requirements



(Figure from: Otazo et al., MRM 2015)

Example (2): Cardiac cine imaging

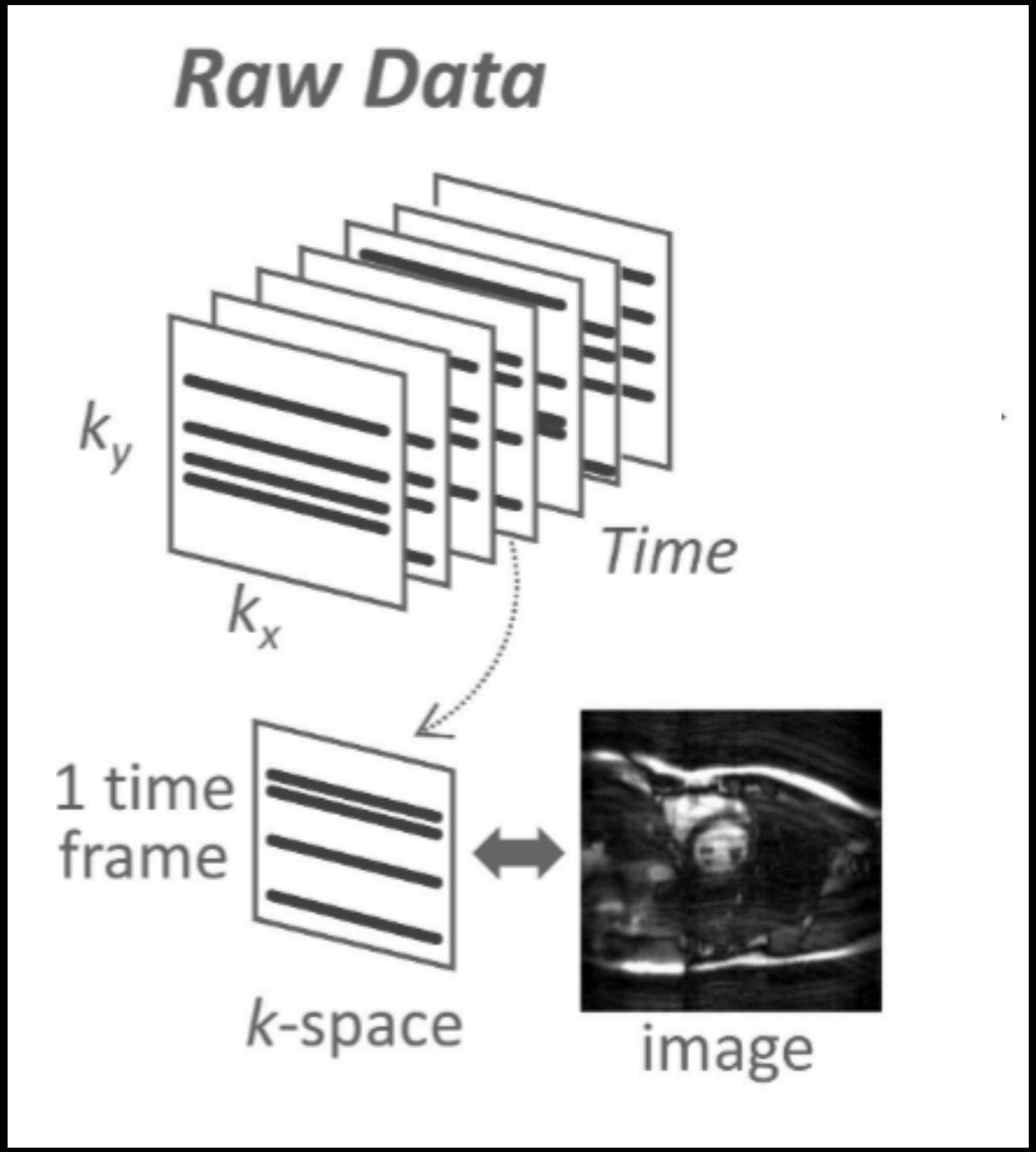
- Sparsity in the x-f space



(Figures from: Tsao et al., JMRI 2012)

Example (2): Cardiac cine imaging

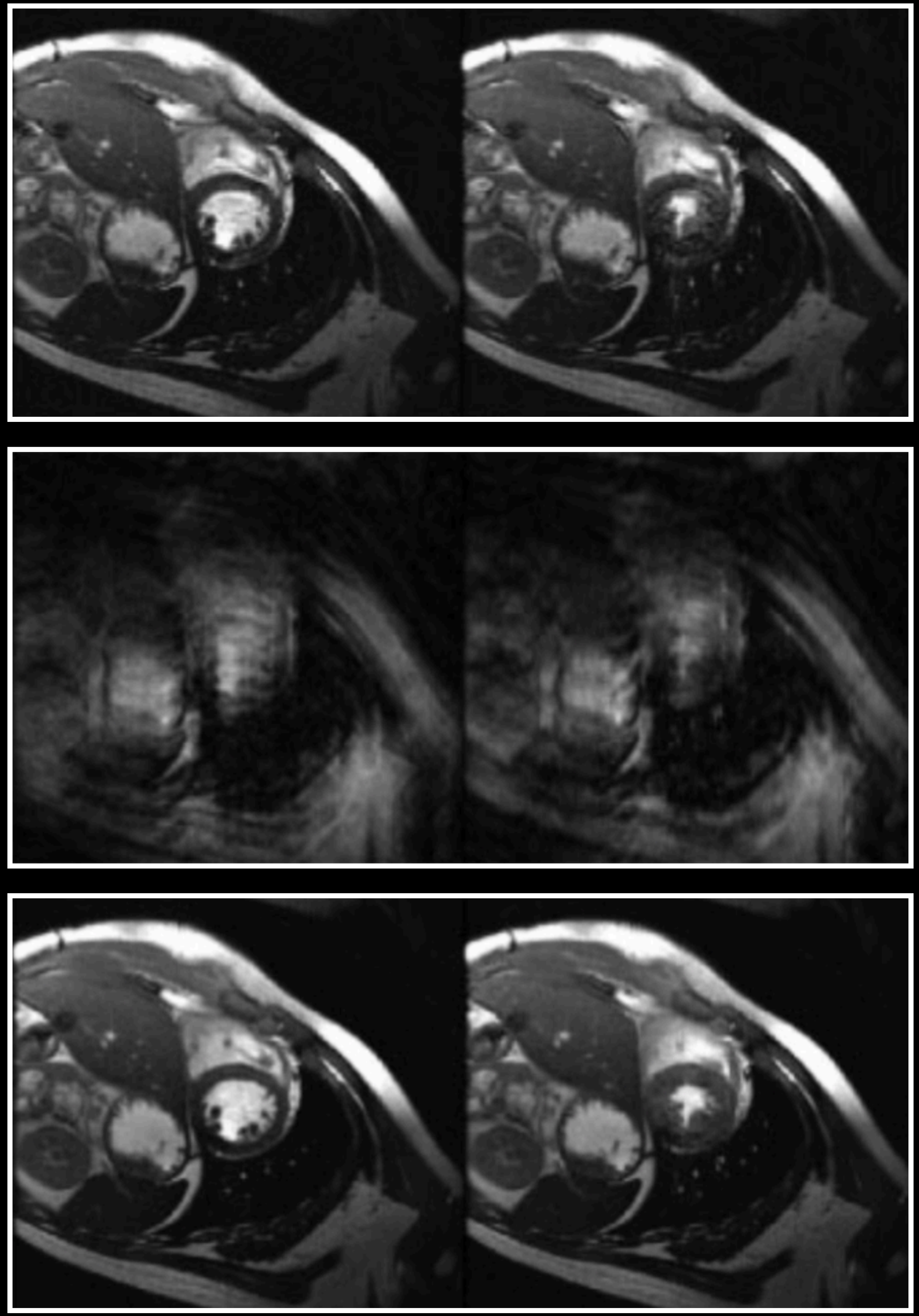
k-t sampling pattern



Fully sampled

6x acceleration with zero-padding

k-t FOCUSS results



(Figures from: Tsao et al., JMRI 2012 and Jung et al., MRM 2009)

Example (2): Cardiac cine imaging

- k-t FOCUSS[1] (*k-t FOCal Underdetermined System Solver*)
  - *Application*: cardiac cine imaging
  - *Constraint*: sparsity in the x-f space
  - *Data sampling*: k-t undersampling
  - *Optimization problem*: $min_\rho \parallel y - DFS\rho \parallel_2^2 + \lambda \parallel \rho \parallel_1$

Let $\rho = \rho_0 + \Delta\rho$

$$min_\rho \parallel y - DFS(\rho_0 + \Delta\rho) \parallel_2^2 + \lambda \parallel \Delta\rho \parallel_1$$

  - *Reconstruction*: reweighted quadratic optimization

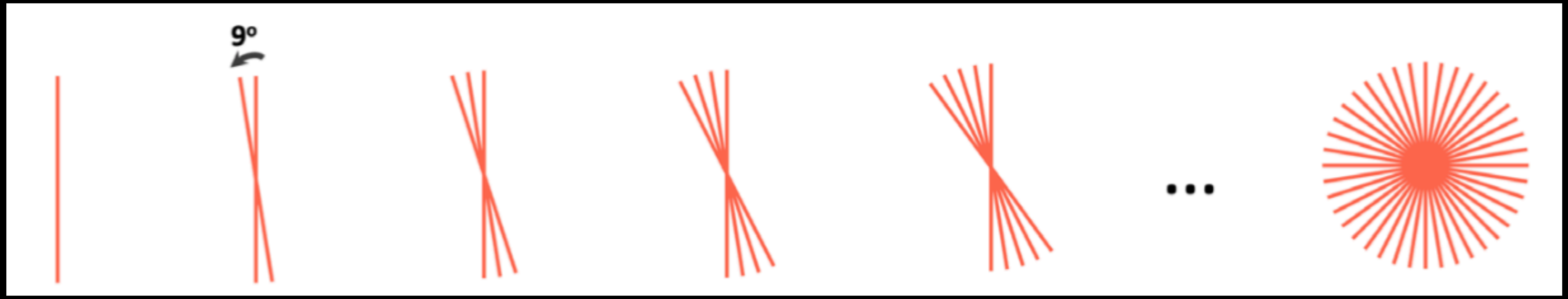
y: acquired k-space data
D: k-t sampling pattern
F: Transform operator between k-space and x-f space
S: coil sensitivity maps
ρ: reconstructed x-f space
λ: regularization parameter

[1] Jung et al., Magn Reson Med. 2009;61(1):103-16

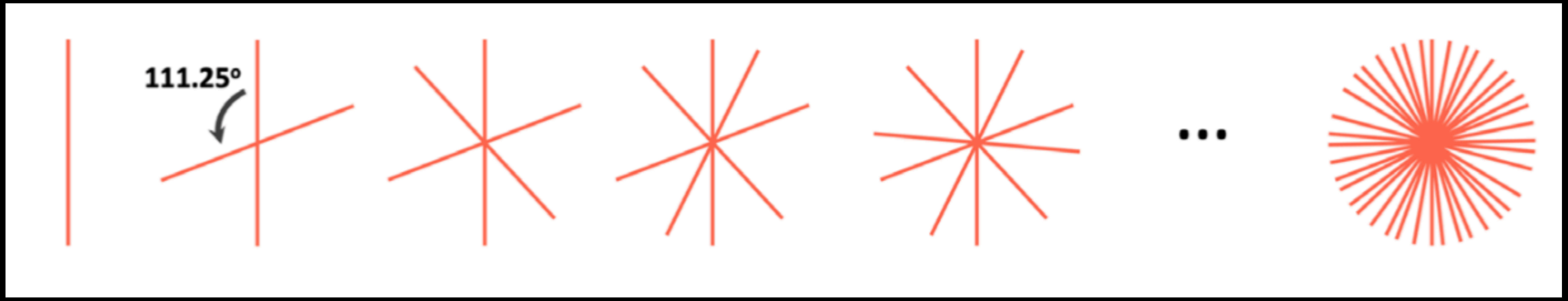
Example (3): Free-breathing radial MRI

- Radial MRI with inherent motion robustness can be used for free-breathing MRI
- Radial undersampling results in incoherent artifacts

Linear radial MRI



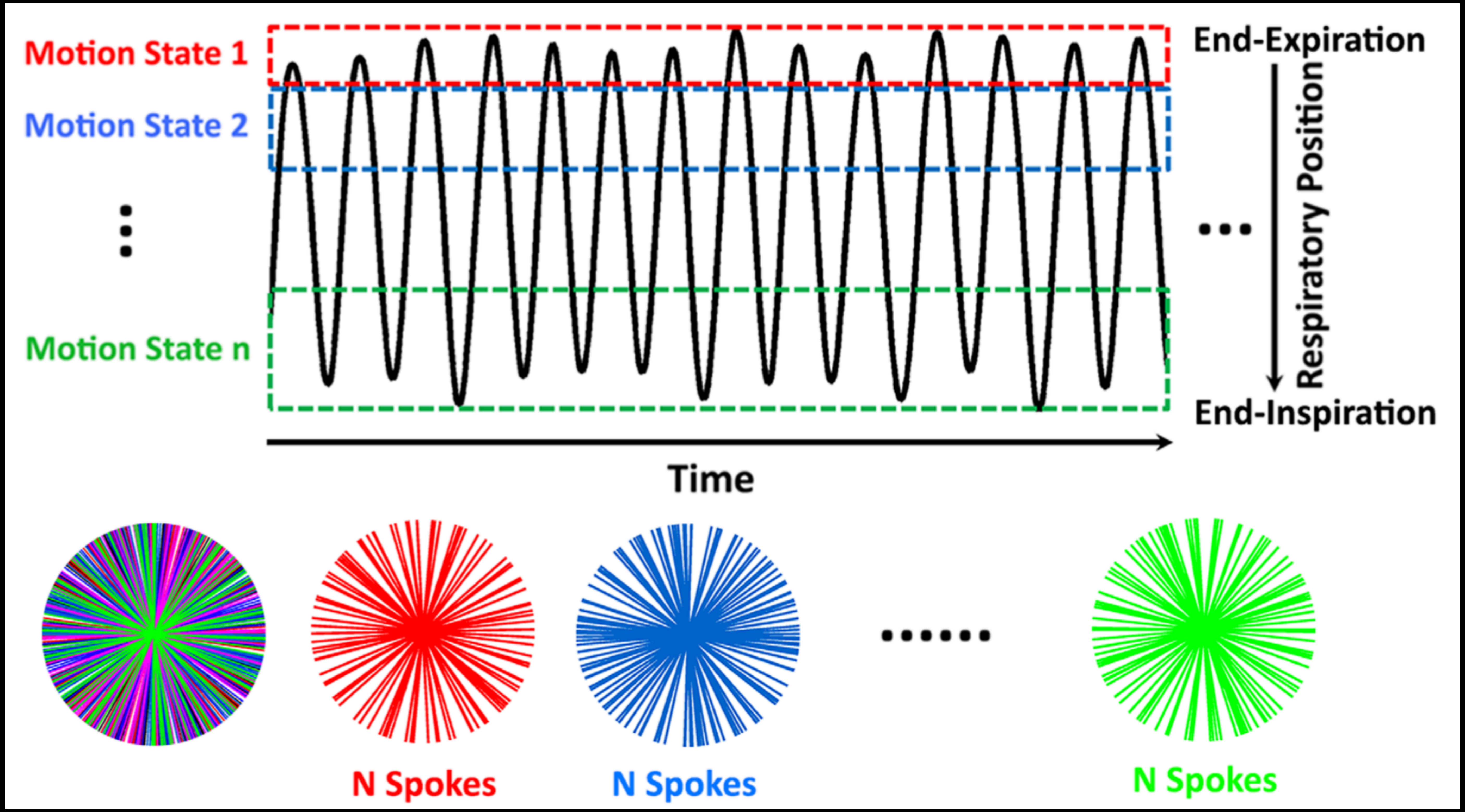
Golden-angle radial MRI



(Figure from: Feng et al., JMRI 2022)

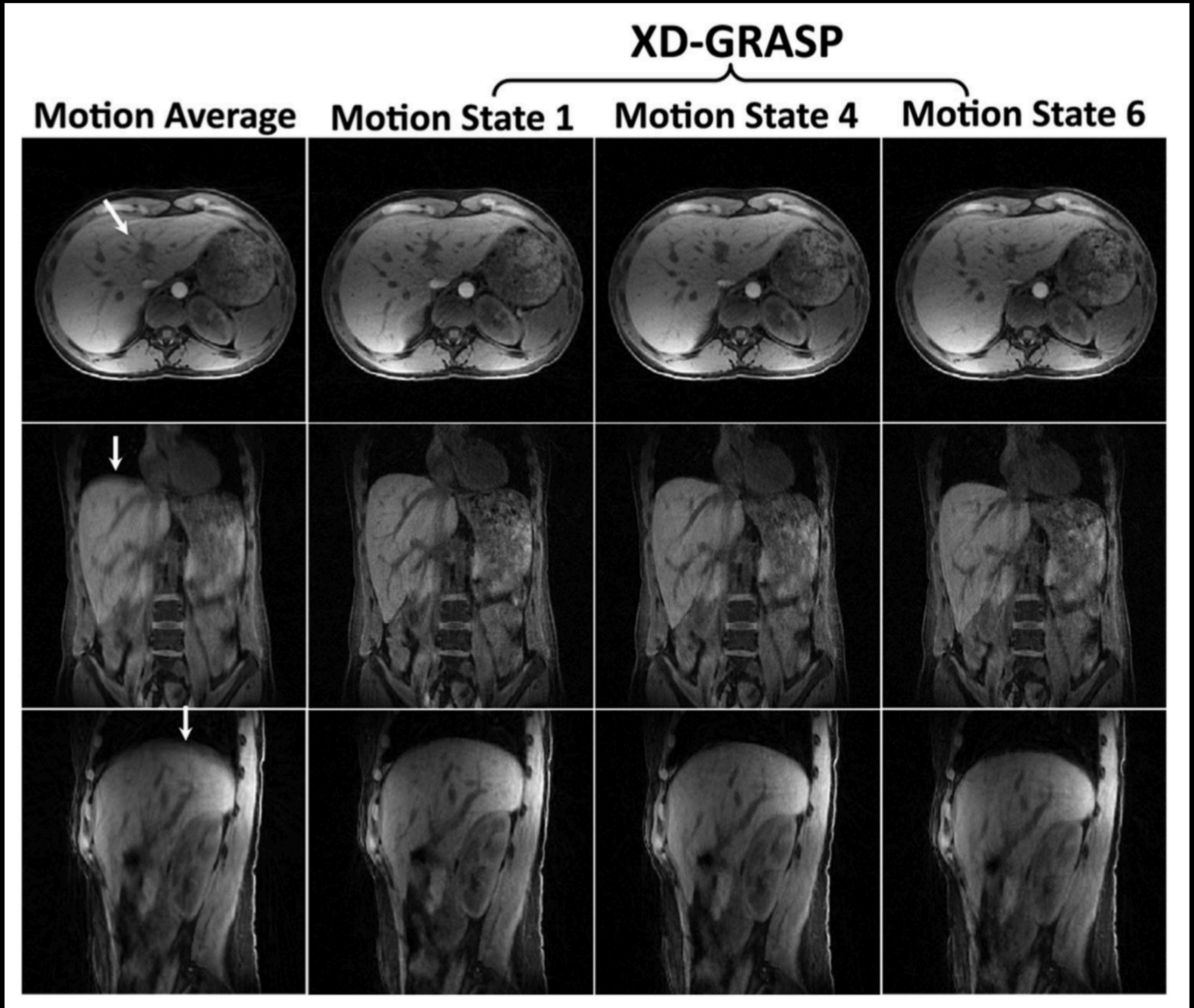
Example (3): Free-breathing radial MRI

- Stack-of-radial MRI provides self-navigation to track breathing motion
- We can group the k-space data into different motion states



(Figure from: Feng et al., MRM 2016)

Example (3): Free-breathing radial MRI



(Figure from: Feng et al., MRM 2016)

Example (3): Free-breathing radial MRI

- XD-GRASP[1] (*Golden-angle radial MRI with reconstruction of extra motion-state dimensions using compressed sensing*)
  - *Application*: free-breathing abdominal imaging
  - *Constraint*: temporal finite differences (or total variation) in dynamic dimension
  - *Data sampling*: undersampled golden-angle radial MRI
  - *Optimization problem*: $min_x \quad \| FCx - y \|_2^2 + \lambda_1 \| S_1 x \|_1 + \lambda_2 \| S_2 x \|_1$
  - *Reconstruction*: non-linear conjugate gradient

[1] Feng et al., Magn Reson Med. 2016;75(2):775-88

Compressed sensing MRI

- Limitations:
  - Requiring high computational complexity to solve the nonlinear reconstruction problem
  - Reconstruction result is dependent on the choice of regularization parameters
  - Reconstruction may fail if the requirements are not met

Compressed sensing MRI

- Conventional compressed sensing MRI requires a pre-determined sparsifying transform (e.g., Wavelet transform) for image reconstruction.
- The assumed sparsity model might not work well in certain applications.

- Attempts to move beyond this limitation…
  - Dictionary-based compressed sensing MRI: Using a learned dictionary of basis functions instead of a specific transform
  - Low-rank based reconstruction: Use the inherent redundancy and low-rank properties in (high-dimensional) MRI dataset for reconstruction
  - Deep learning-based reconstruction: Use the information learned from the large datasets to reconstruct undersampled MRI data

Take home message

- 3 main components for compressed sensing MRI to work
  - The image has a sparse representation in some transform domain
  - The k-space sampling trajectory generates incoherent artifacts in the sparse transform domain
  - It involves a nonlinear reconstruction method

Take home message

- If we want to apply compressed sensing to accelerate an MRI application, check:
- (1) Can the images be sparsified in a certain (transform) domain?
  - Wavelet transform
  - Spatial total variation in images
  - Total variation in temporal frames
  - x-f space
  - …
- (2) Can the sampling pattern generate incoherent undersampling artifacts?
  - Variable density sampling pattern
  - Radial acquisition
  - Spiral acquisition
  - …

Thanks!

- Next lecture
  - Fast Imaging - Non-Cartesian Sampling by Dr. Wu

- See you next time
  - Deep learning MRI Reconstruction on 5/22

Questions?

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