Image Reconstruction

Parallel Imaging

M229 Advanced Topics in MRI

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Fourier Transform Symmetry

$$F(f) = \int_{-\infty}^{\infty} f_e(x) \cos(2\pi x f) dx - j \int_{-\infty}^{\infty} f_o(x) \sin(2\pi x f) dx$$

$$F(f) = F_e(f) + F_o(f)$$

real & even function?

real & odd function?

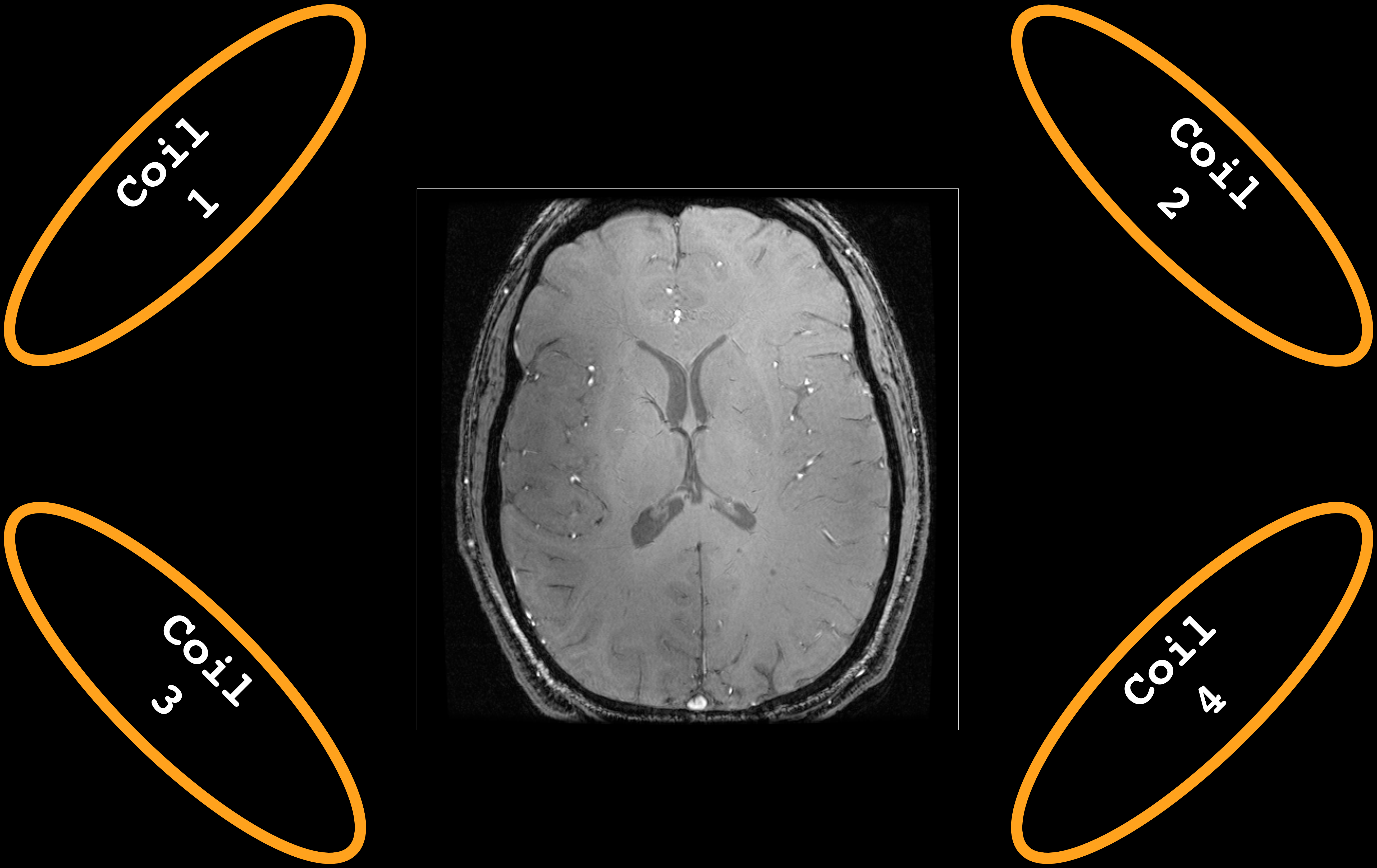
even function?

odd function?

Today's Topics

- Multicoil reconstruction
- Parallel imaging
  - Image domain methods:
    - SENSE
  - k-space domain methods:
    - SMASH
    - GRAPPA

Multi-coil Arrays



Coil 1

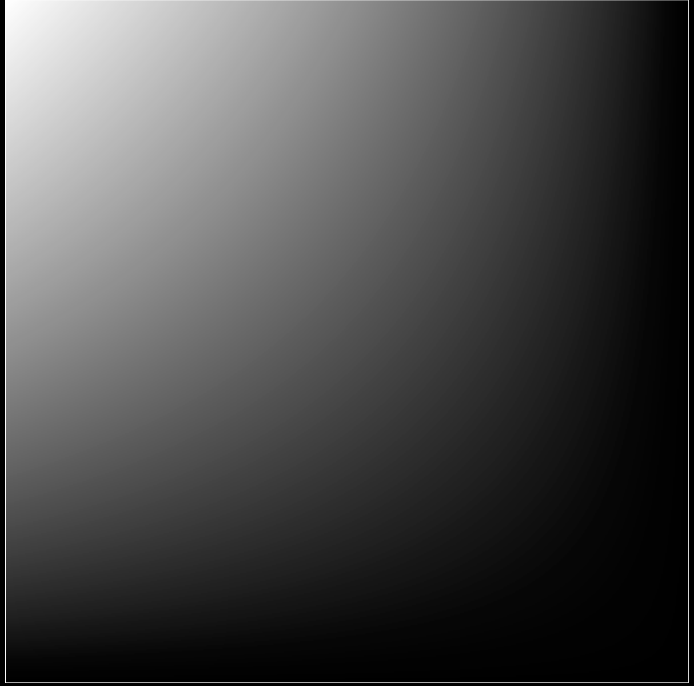
Coil 2

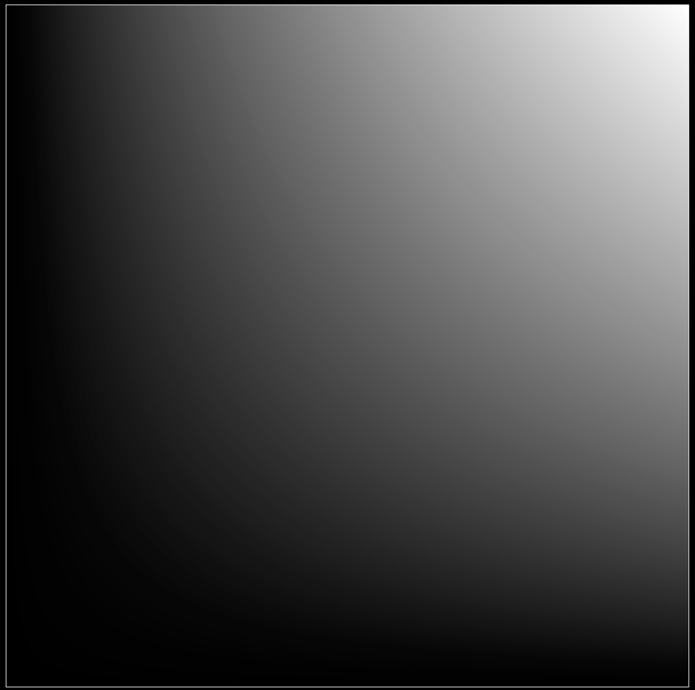
Coil 3

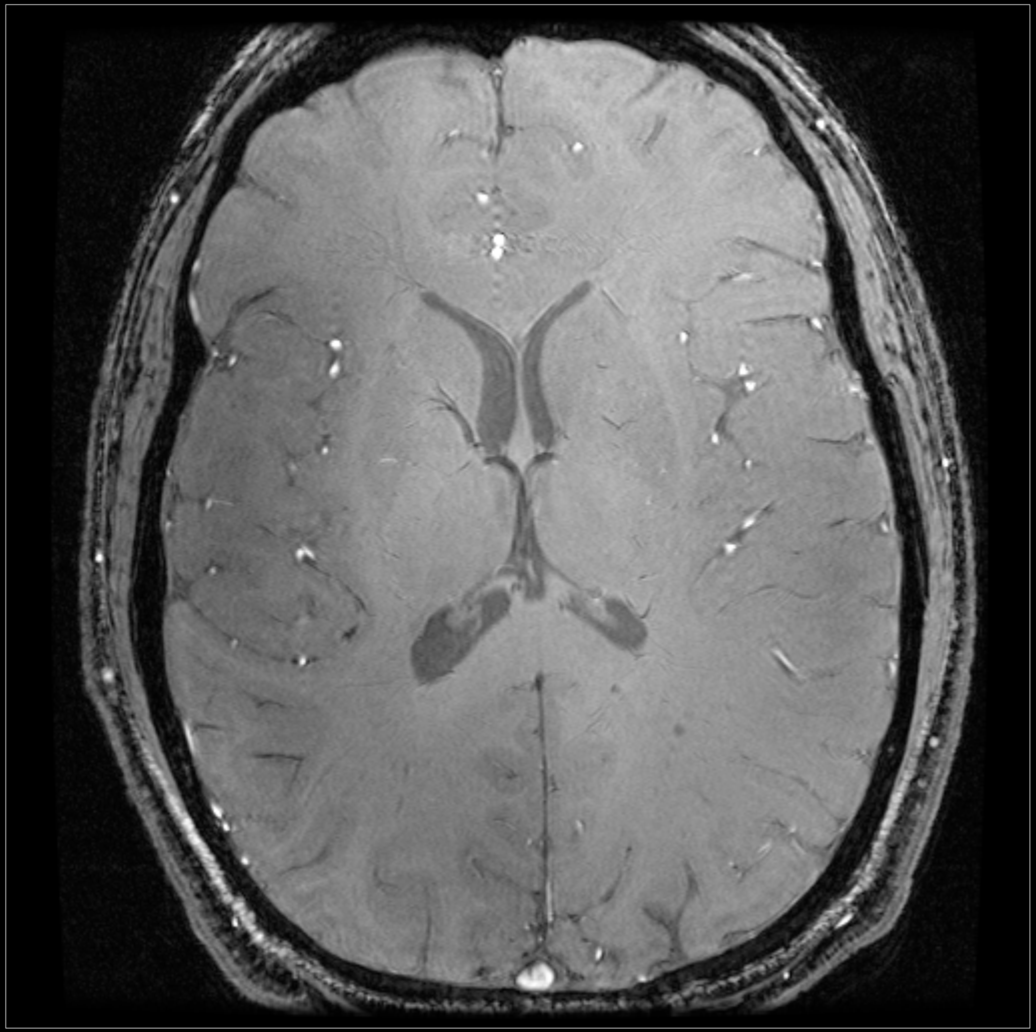
Coil 4

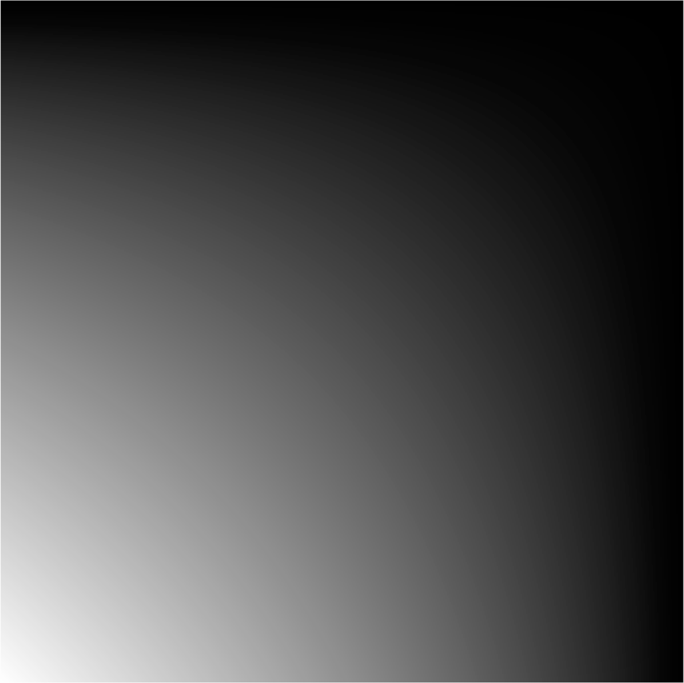
Multi-coil Sensitivity

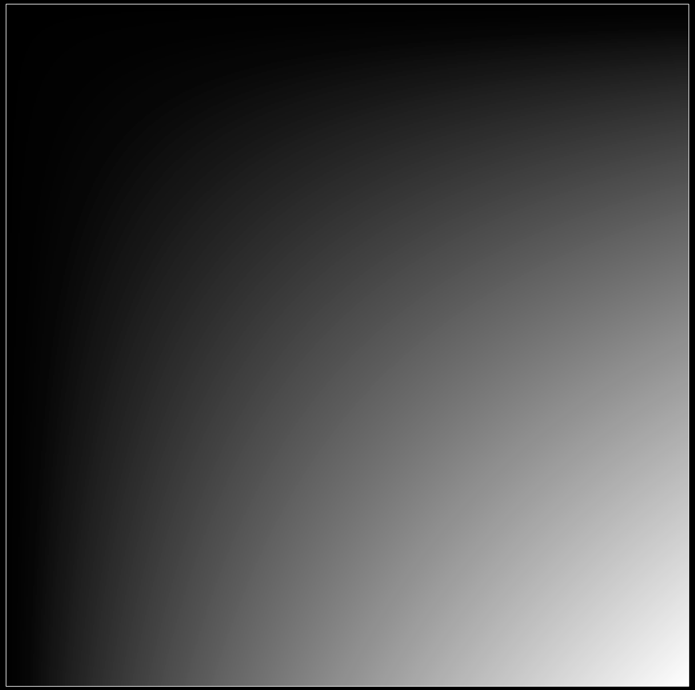
$$\left\| \vec{B}\left(\vec{r}\right) \right\|$$











Multi-coil Reconstruction

- Each coil has a complete image of whole FOV and an amplitude and phase sensitivity

$$C_l(\vec{x}) \qquad l = 1,2, \ldots L$$

- Coils are coupled, so noise is correlated

$$E[n_i n_j] = \Psi$$

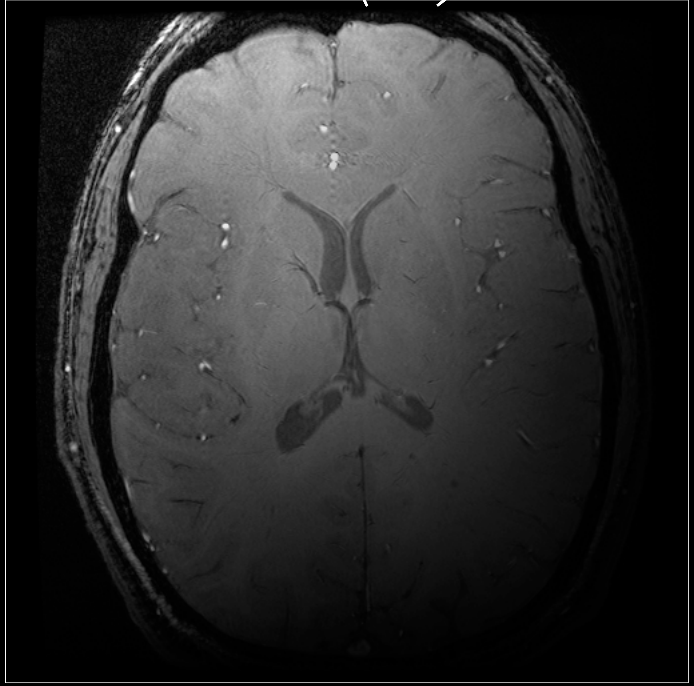
- Received data from coil l:

$$m_l(\vec{x}) = C_l(\vec{x}) m(\vec{x}) + n_l(\vec{x})$$

- Given $m_l(x)$, how do we reconstruct $m(x)$?

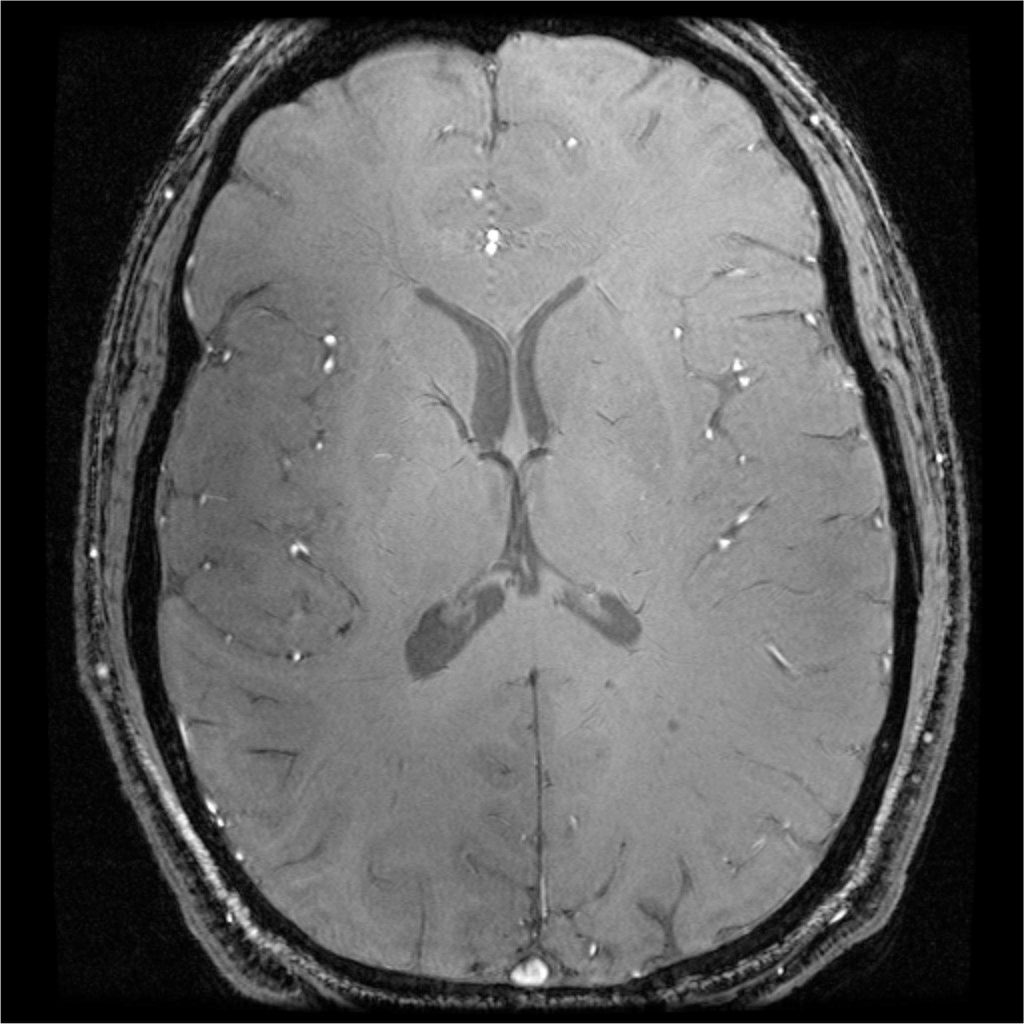
Multi-coil Images

$m_1(x)$

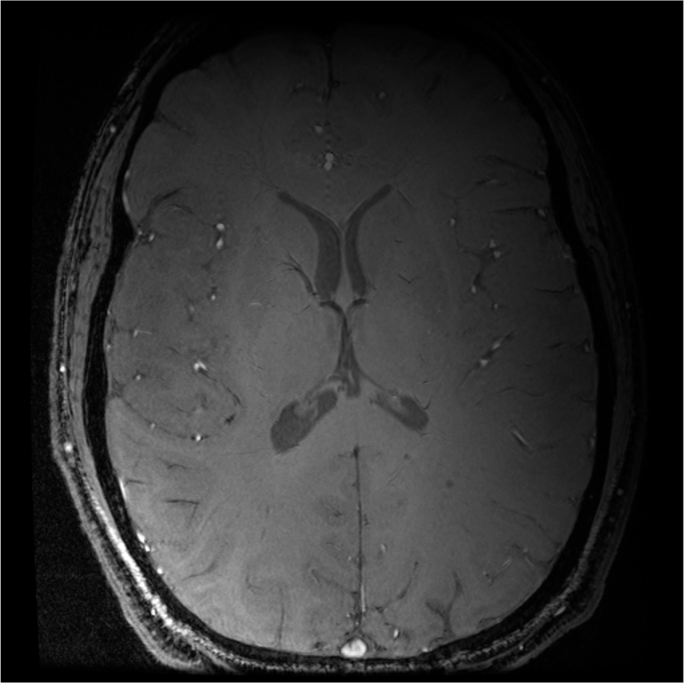


$m_2(x)$

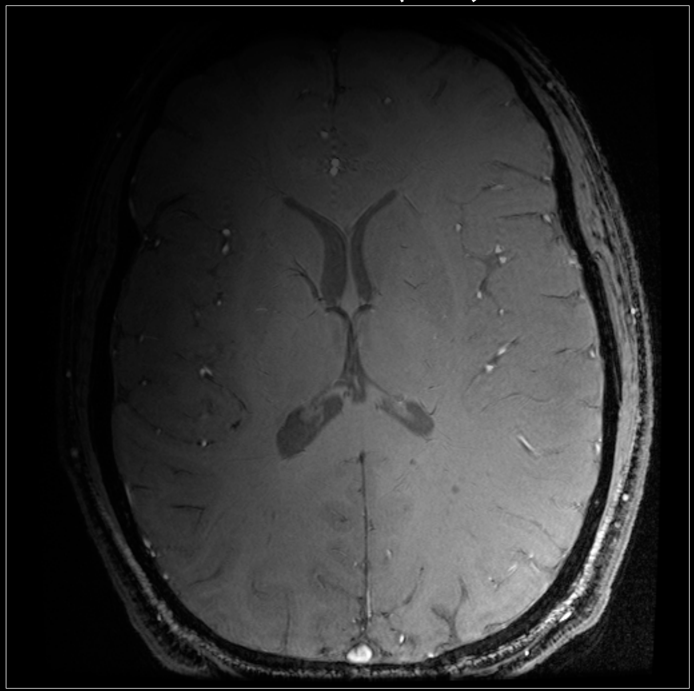
$m_s(x)$



$m_3(x)$



$m_4(x)$



Multi-coil Reconstruction

For a particular voxel x

$$\begin{pmatrix} m_1(\vec{x}) \\ m_2(\vec{x}) \\ \cdot \\ \cdot \\ \cdot \\ m_L(\vec{x}) \end{pmatrix} = \begin{pmatrix} C_1(\vec{x}) \\ C_2(\vec{x}) \\ \cdot \\ \cdot \\ \cdot \\ C_L(\vec{x}) \end{pmatrix} m(\vec{x}) + \begin{pmatrix} n_1(\vec{x}) \\ n_2(\vec{x}) \\ \cdot \\ \cdot \\ \cdot \\ n_L(\vec{x}) \end{pmatrix}$$

OR

$$\underset{\text{L x 1}}{\underline{m_s(\vec{x})}} = \underset{\text{L x 1}}{\underline{C}}\, m(\vec{x}) + \underset{\text{L x 1}}{\underline{n}}$$

Minimum Variance Estimate

$$\hat{m}(\vec{x}) = \underbrace{(C^*\Psi^{-1}C)^{-1}}_{\text{1 x 1}}\underbrace{C^*\Psi^{-1}}_{\text{1 x L}}\underbrace{m_s(\vec{x})}_{\text{L x 1}}$$

Covariance (variance)

$$COV(\hat{m}(\vec{x})) = C^*\Psi^{-1}C$$

What if Ψ is $\sigma^2\mathrm{I}$?

$$\hat{m}(\vec{x}) = \underbrace{(C^*C)^{-1}}_{\substack{\text{Intensity}\\ \text{Correction}}}\underbrace{C^*}_{\substack{\text{Phase}\\ \text{Correction}}}m_s(\vec{x})$$

Approximate Solution

- Often we don't know $C_l(x)$, but

$$m_l(\vec{x}) = C_l(\vec{x})m(\vec{x})$$

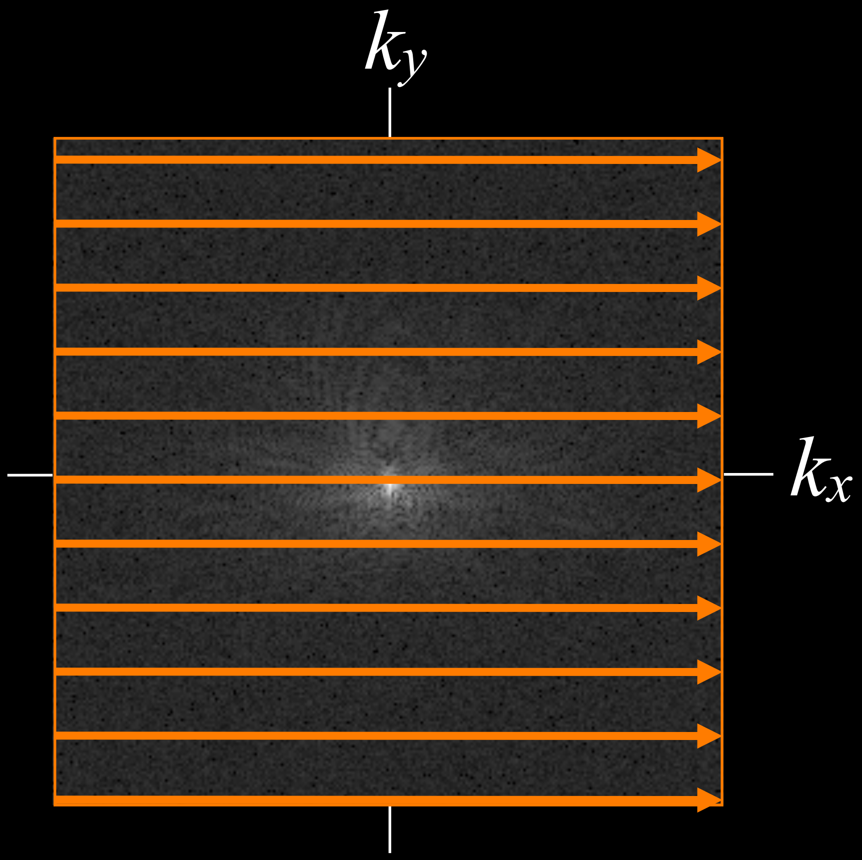
- Approximate solution:

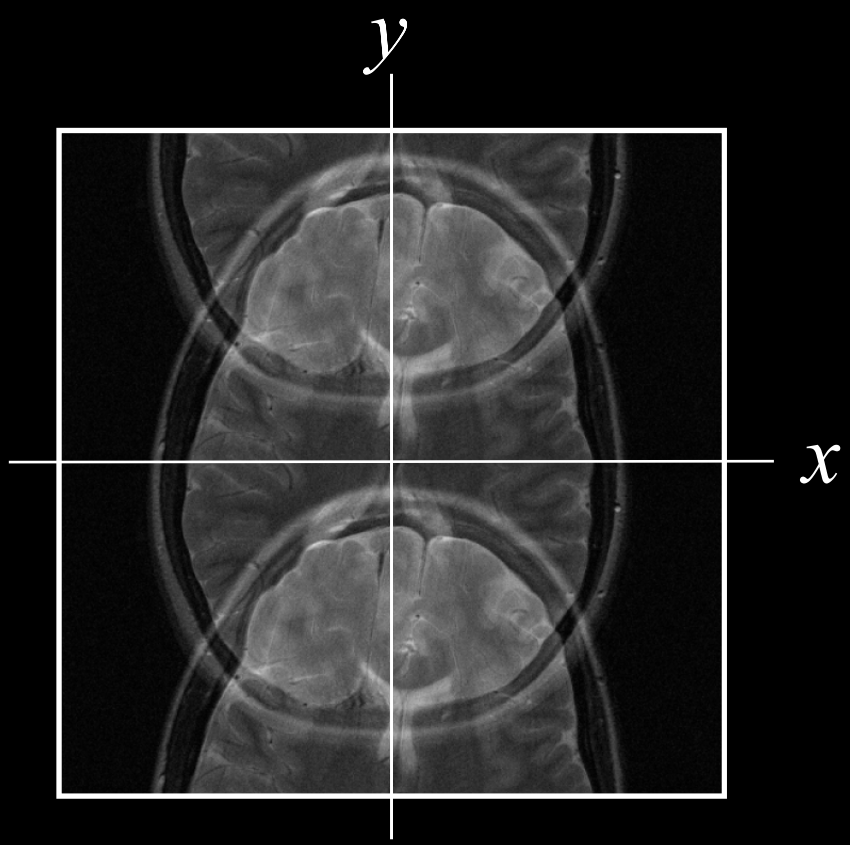
$$\hat{m}_{SS}(\vec{x}) = \sqrt{\sum_l m_l^*(\vec{x})m_l(\vec{x})}$$

- For SNR > 20, within 10% of optimal solution

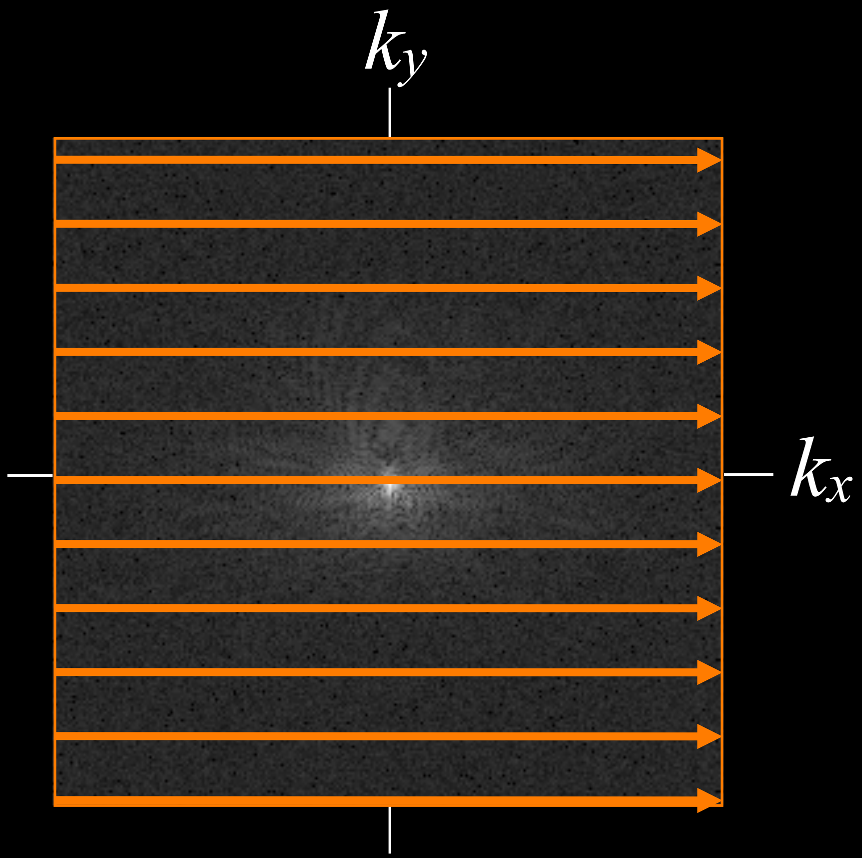
PB Roemer et al. MRM 1990

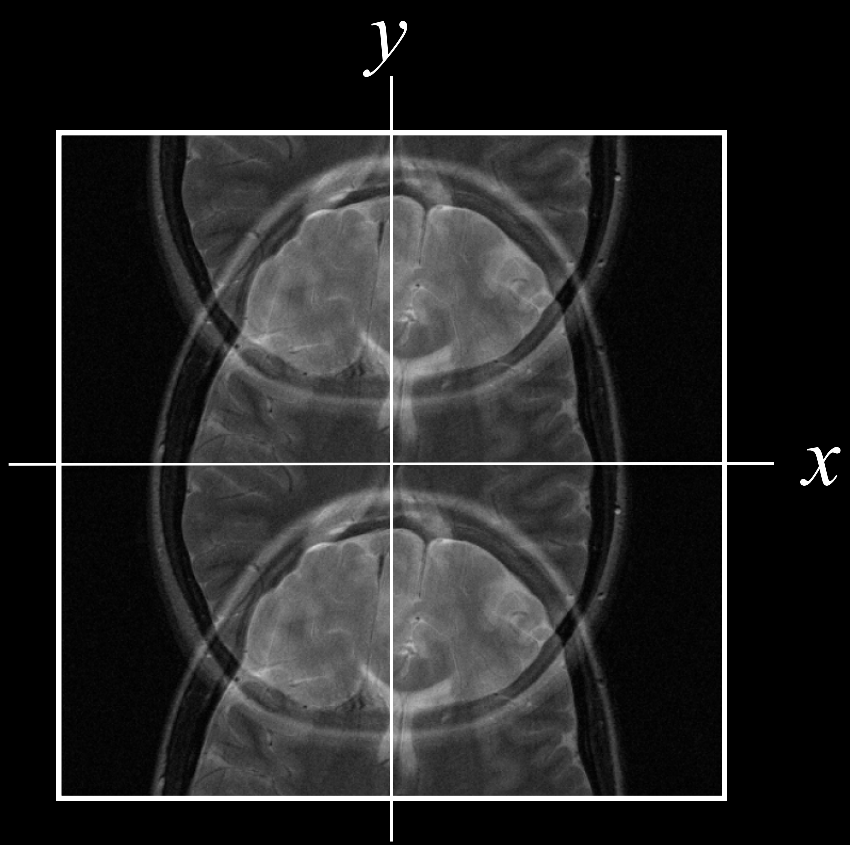
Accelerate Imaging with Array Coils





Accelerate Imaging with Array Coils





- Parallel Imaging
  - Coil elements provide some localization
  - Undersample in k-space, producing aliasing
  - Sort out in reconstruction

Parallel Imaging

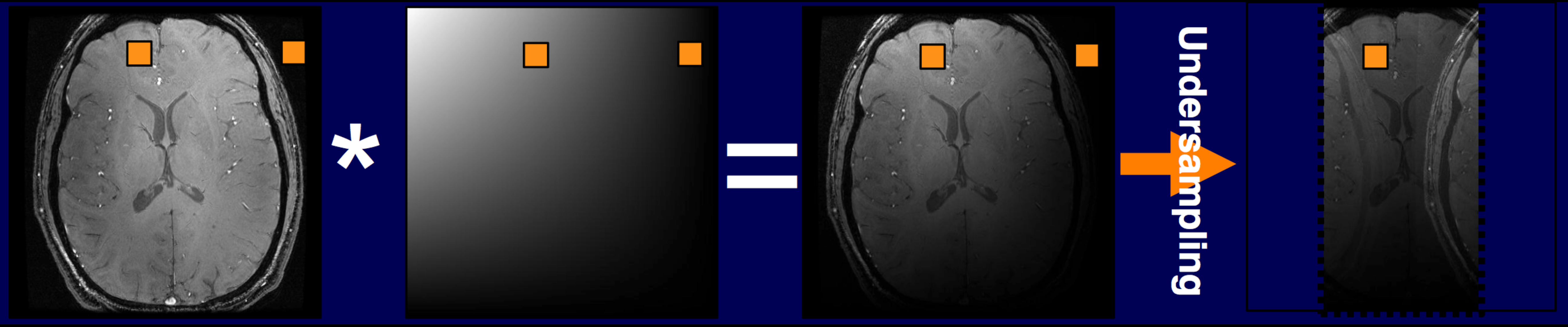
- Many approaches:
  - Image domain - SENSE
  - k-space domain - SMASH, GRAPPA
  - Hybrid - ARC

- We will focus on two:
  - SENSE: optimal if you know coil sensitivities
  - GRAPPA: autocalibrating / robust

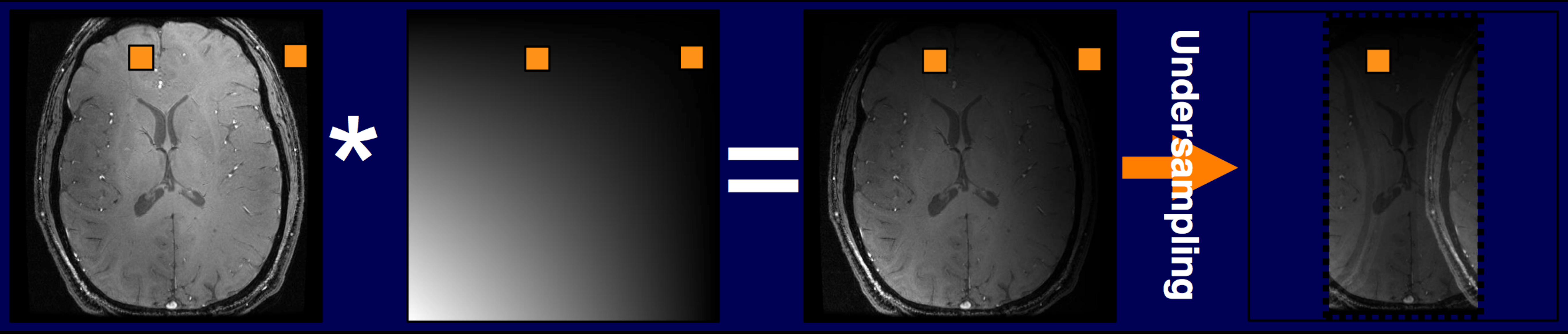
Parallel Imaging
(SENSE)

Cartesian SENSE

$$m_1(\vec{x_1}) = C_1(\vec{x_1})m(\vec{x_1}) + C_1(\vec{x_2})m(\vec{x_2})$$



$$m_2(\vec{x_1}) = C_2(\vec{x_1})m(\vec{x_1}) + C_2(\vec{x_2})m(\vec{x_2})$$



$$\underbrace{\begin{pmatrix} m_1(\vec{x_1}) \\ m_2(\vec{x_1}) \\ \cdot \\ \cdot \\ \cdot \\ m_L(\vec{x_1}) \end{pmatrix}}_{\text{Aliased Images}} = \underbrace{\begin{pmatrix} C_1(\vec{x_1}) & C_1(\vec{x_2}) \\ C_2(\vec{x_1}) & C_2(\vec{x_2}) \\ \cdot & \\ \cdot & \\ \cdot & \\ C_L(\vec{x_1}) & C_L(\vec{x_2}) \end{pmatrix}}_{\text{Sensitivity at Source Voxels}} \underbrace{\begin{pmatrix} m(\vec{x_1}) \\ m(\vec{x_2}) \end{pmatrix}}_{\text{Source Voxels}} + \begin{pmatrix} n_1(\vec{x_1}) \\ n_2(\vec{x_1}) \\ \cdot \\ \cdot \\ \cdot \\ n_L(\vec{x_1}) \end{pmatrix}$$

OR

$$\underbrace{m_s}_{\text{L x 1}} = \underbrace{C}_{\text{L x 2}} \overset{\text{2 x 1}}{\boxed{m}} + \underbrace{n}_{\text{L x 1}}$$

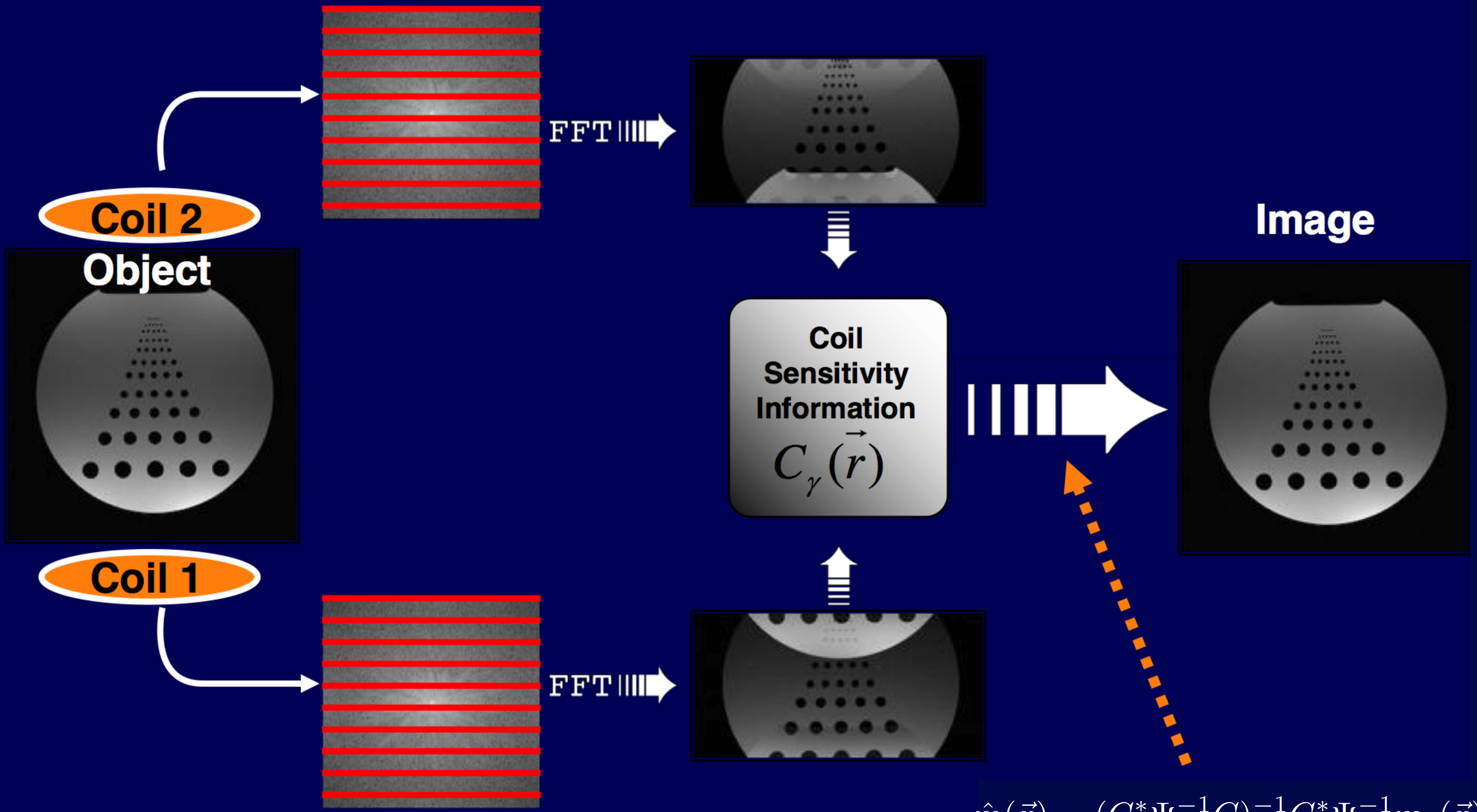
$$\hat{m}(\vec{x}) = {\underbrace{(C^*\Psi^{-1}C)}_{\text{2 x 2}}}^{-1}\underbrace{C^*\Psi^{-1}}_{\text{2 x L}}\underbrace{m_s(\vec{x})}_{\text{L x 1}}$$

L aliased reconstruction resolves 2 image pixels

For an N x N image, we solve (N/2 x N) 2 x 2 inverse systems

For an acceleration factor R, we solve (N/R x N) R x R inverse systems

SENSE Reconstruction



$$\hat{m}(\vec{x}) = (C^* \Psi^{-1} C)^{-1} C^* \Psi^{-1} m_s(\vec{x})$$

Unwrap fold over in image space

SNR Cost

- How large can R be?
- Two SNR loss mechanisms
  - Reduced scan time
  - Condition of the SENSE decomposition
- SNR Loss

$$SNR_{SENSE} = \frac{SNR}{\underset{\text{Geometry Factor}}{\underline{g}}\,\underset{\text{Reduced Scan Time}}{\underline{\sqrt{R}}}}$$

Geometry Factor

- Covariance for a fully sampled image (variance of one voxel):

$$\chi_F = \frac{1}{n_F}(C_F^* \Psi^{-1} C_F)^{-1}$$

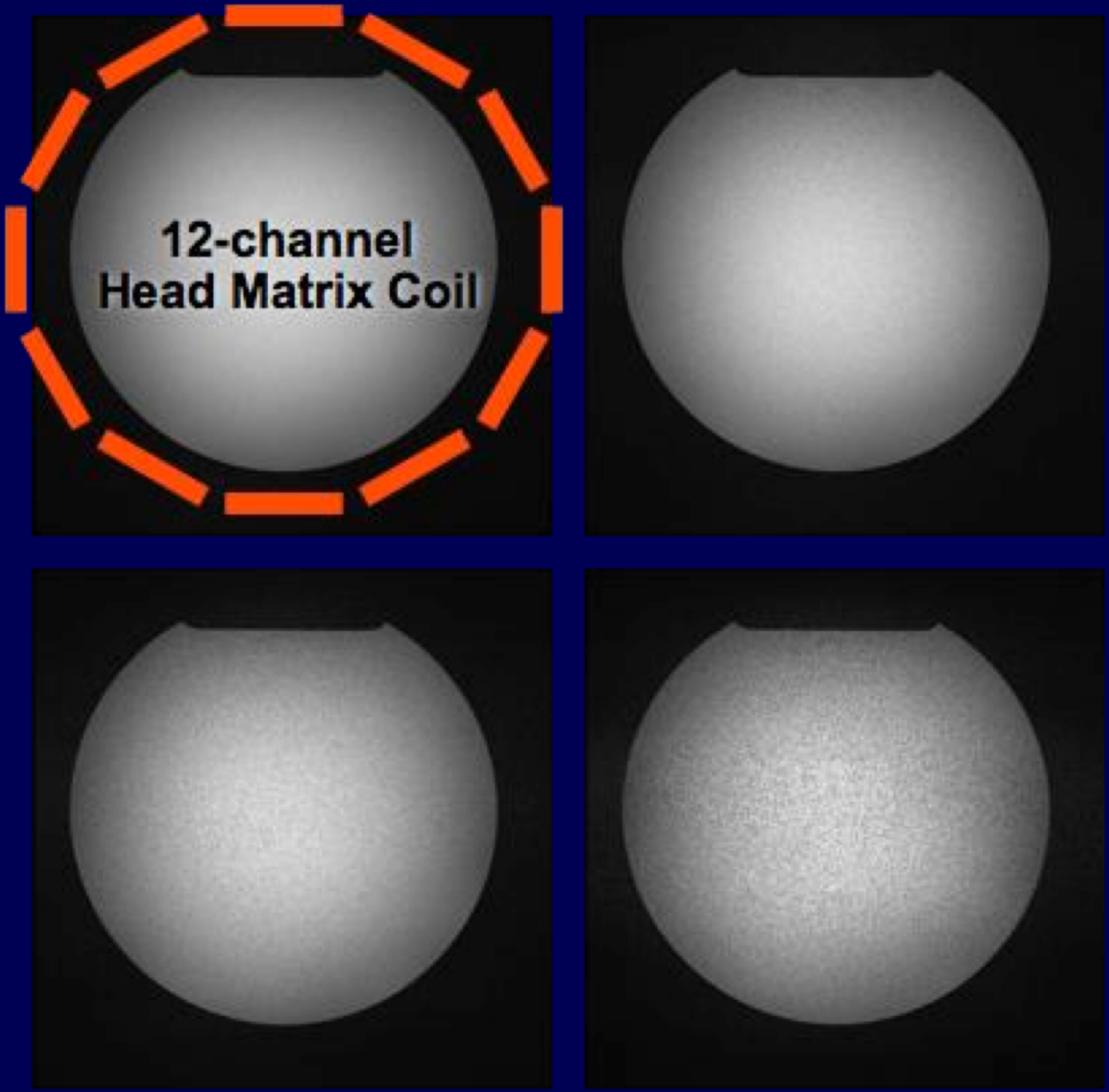
- Covariance for a reduced encoded image:

$$\chi_R = \frac{1}{n_R}(C_R^* \Psi^{-1} C_R)^{-1}$$

Geometry Factor

- g-factor is critical since it depends on:
  - Acceleration
  - Spatial position
  - Aliasing direction
  - Coil geometry
- Minimizing g-factor drives system design
- Sense coils are different from traditional array coils

Parallel Imaging Tradeoffs



ref

PAT x 2

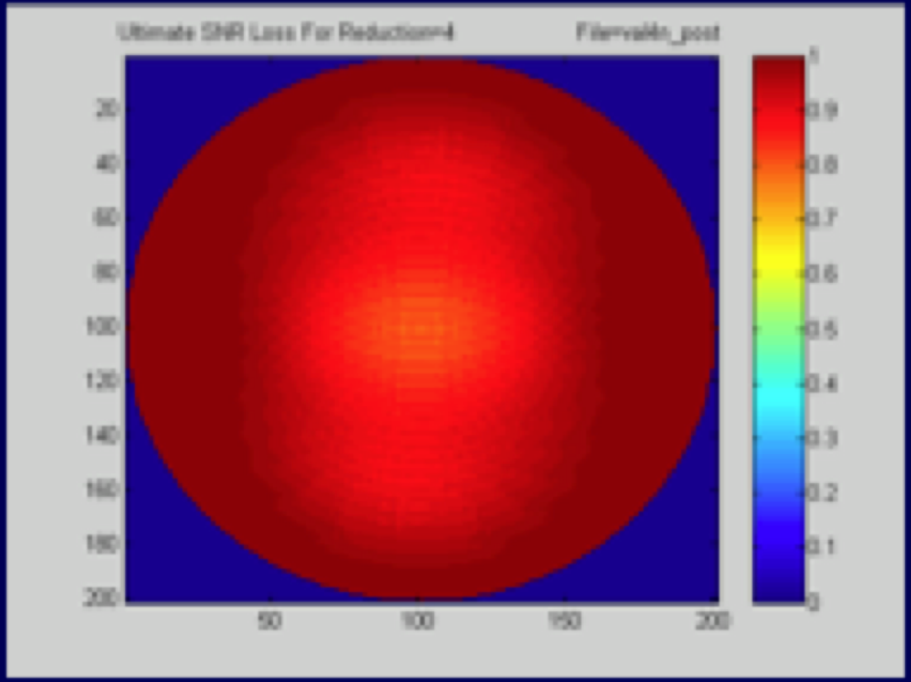
f_p = acceleration factor

g = coil geometry factor

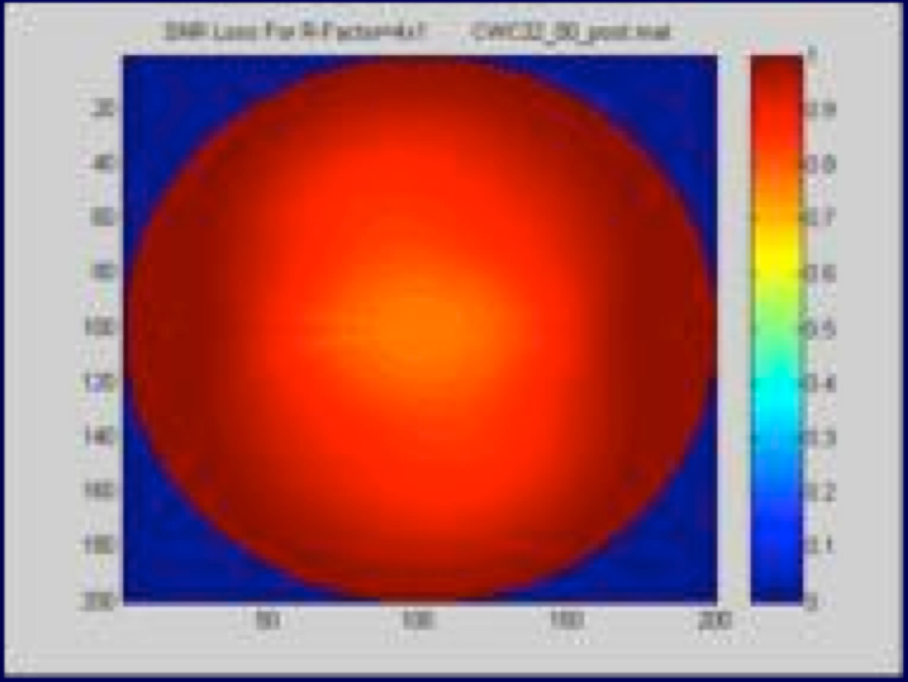
PAT x 3

PAT x 4

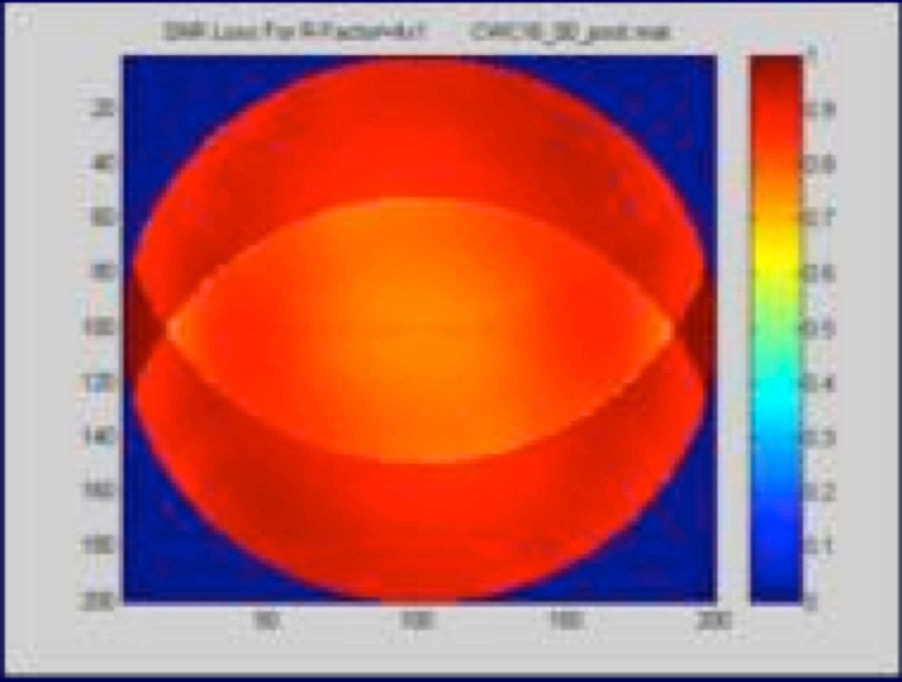
1/g-factor Map for R=4



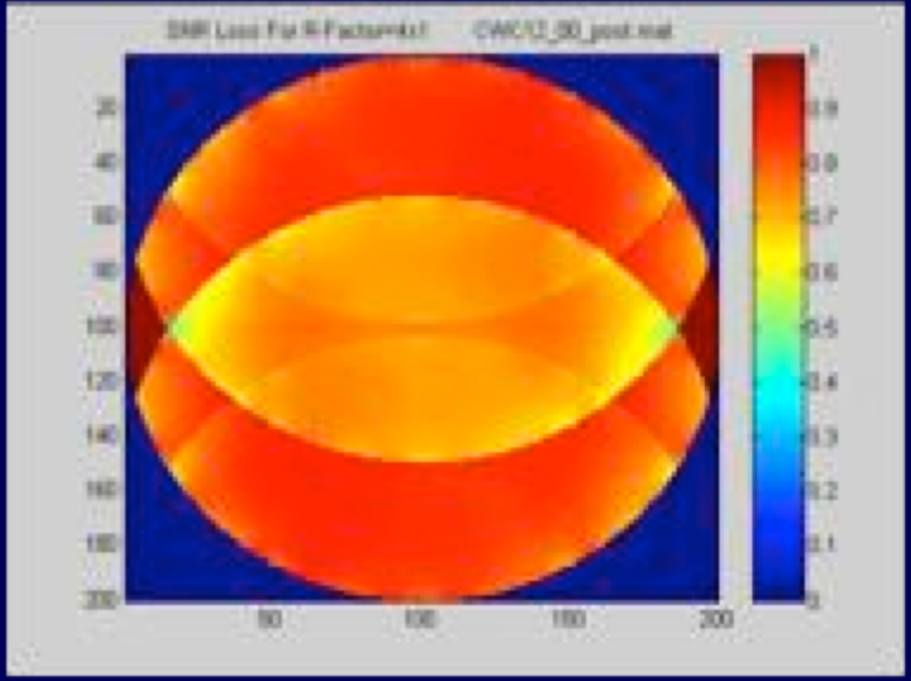
∞ elements



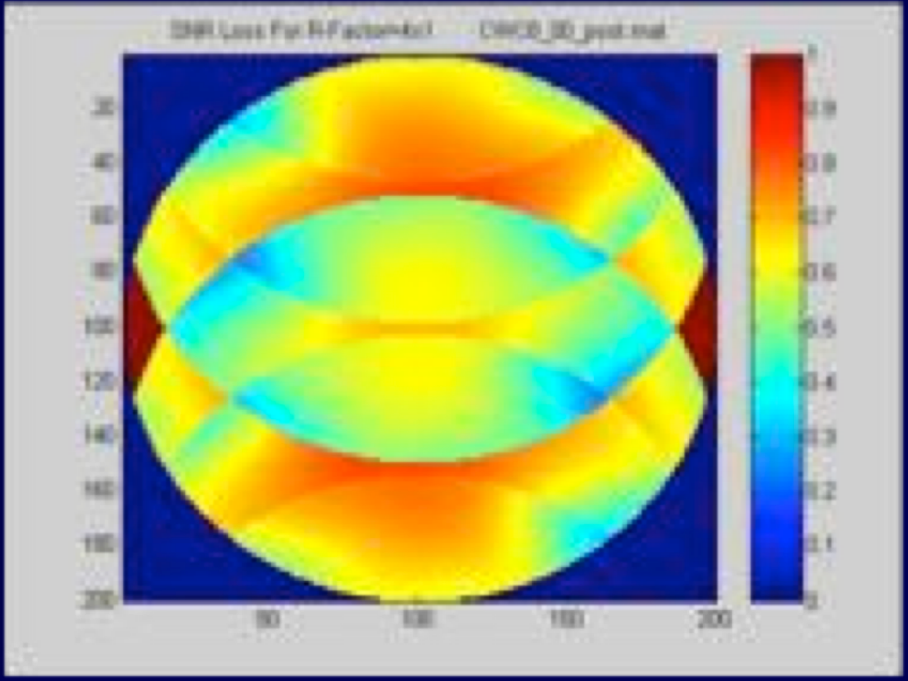
32 elements



16 elements



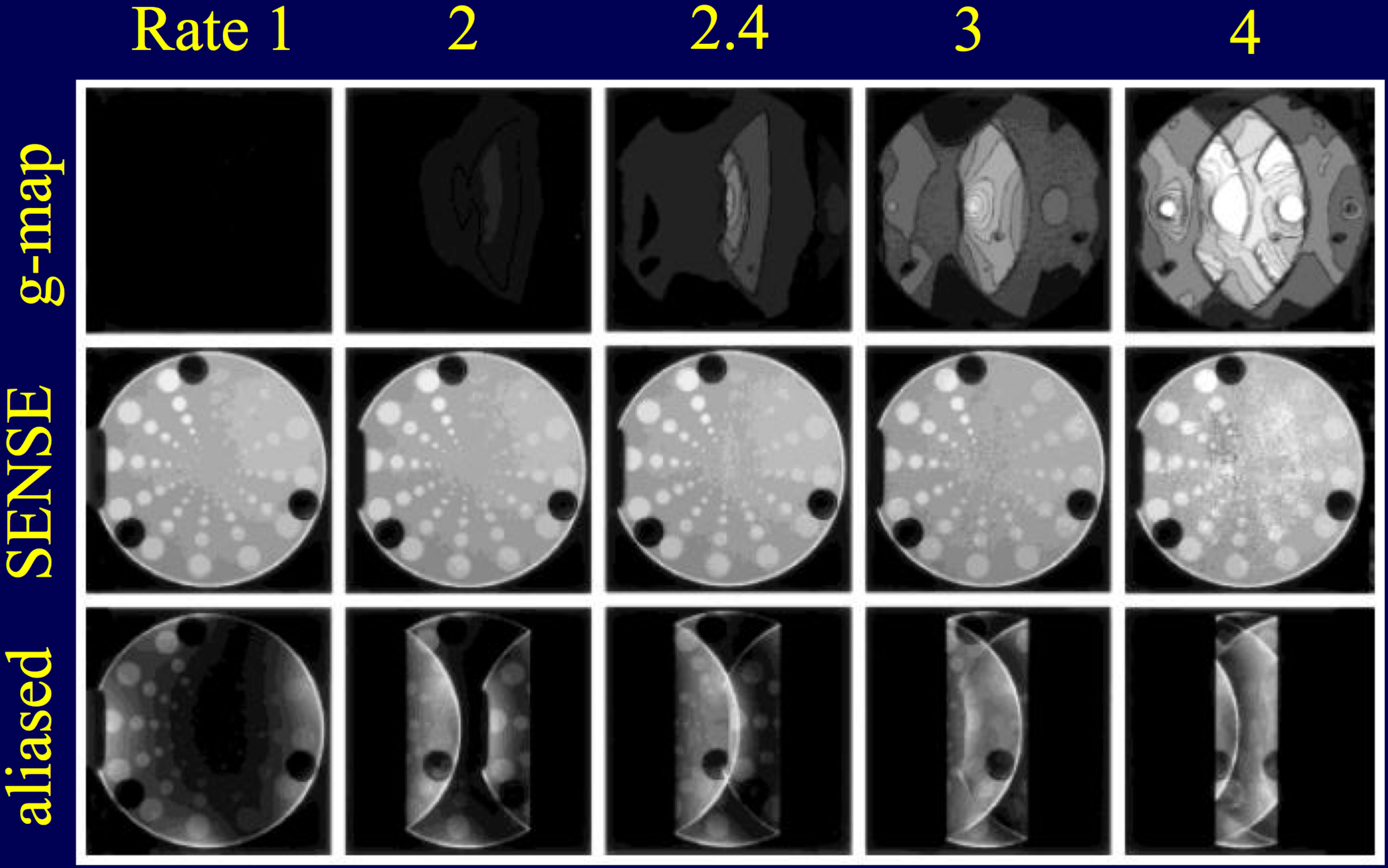
12 elements



8 elements

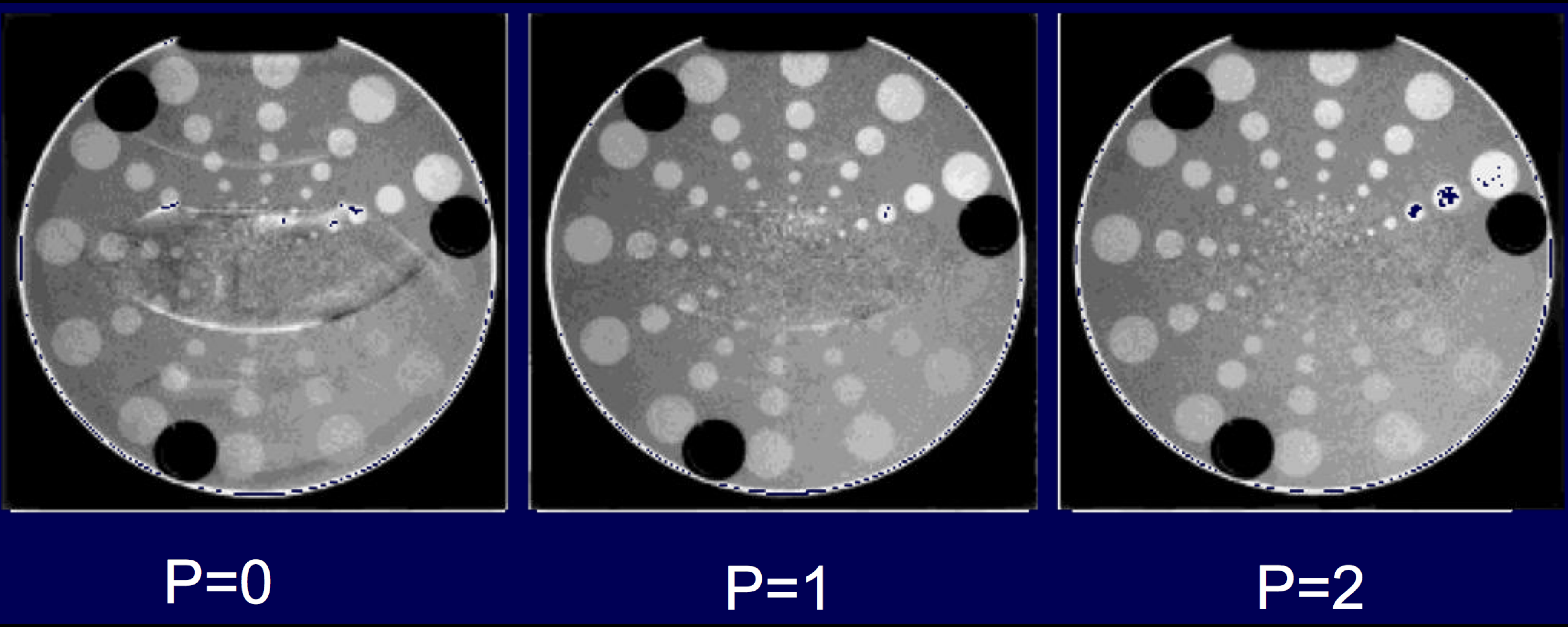
Relative
SNR
Scale

g-factor and its impact on images



Dependence on Coil Sensitivity

- Images reconstructed using coil sensitivity maps with different order P of polynomial fitting

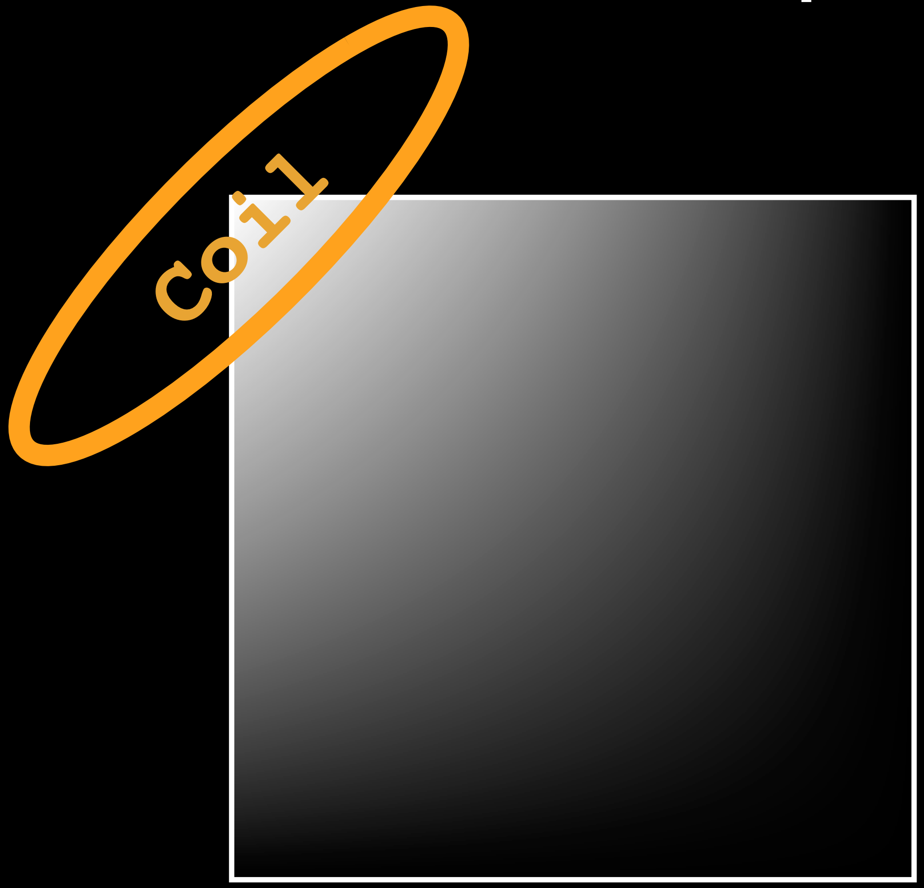


Pruessmann et al. MRM 1999

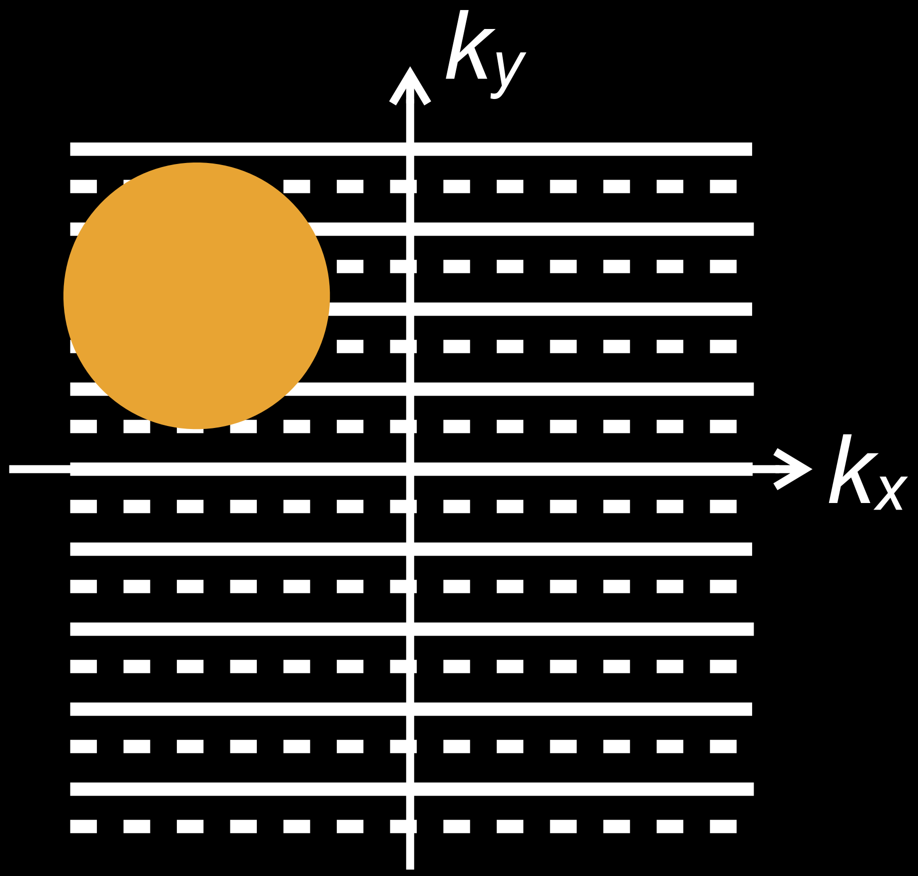
Parallel Imaging
(GRAPPA)

GRAPPA

- Coil sensitivities are
  - Smooth in image space
  - Local in k-space



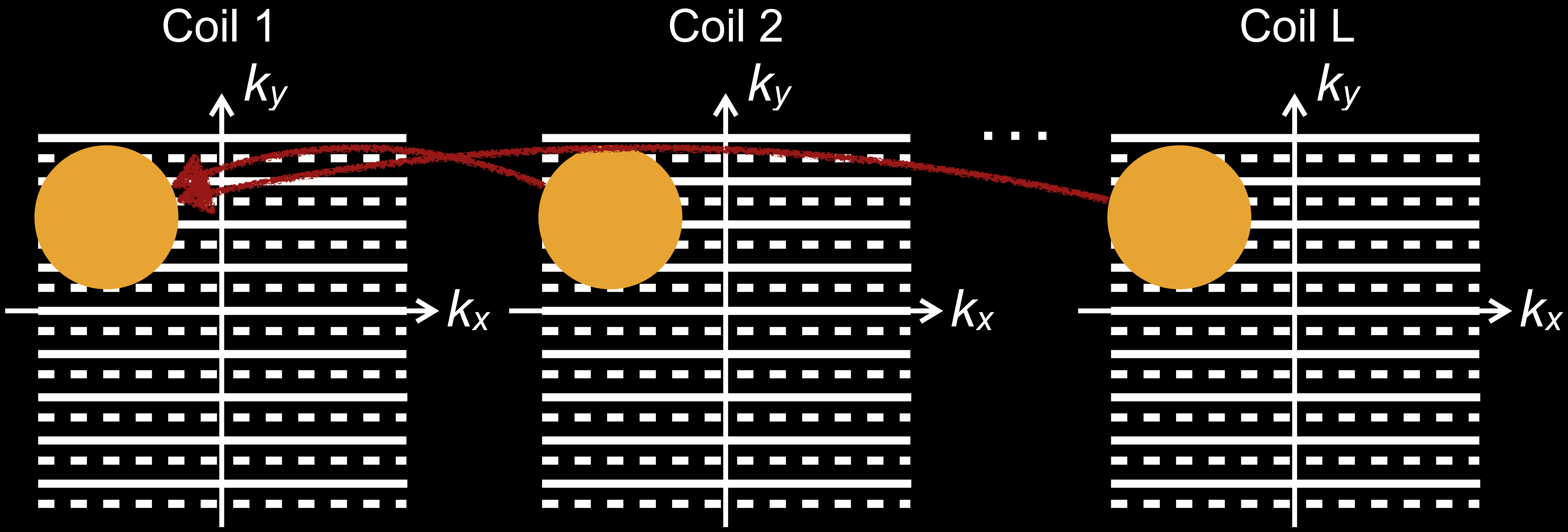
$$m(\vec{x})C_j(\vec{x})$$



$$M(\vec{k}) * C_j(\vec{k})$$

GRAPPA

- Missing information is implicitly contained by adjacent data



$M(\vec{k}) * C_1(\vec{k})$ $M(\vec{k}) * C_2(\vec{k})$ $M(\vec{k}) * C_L(\vec{k})$

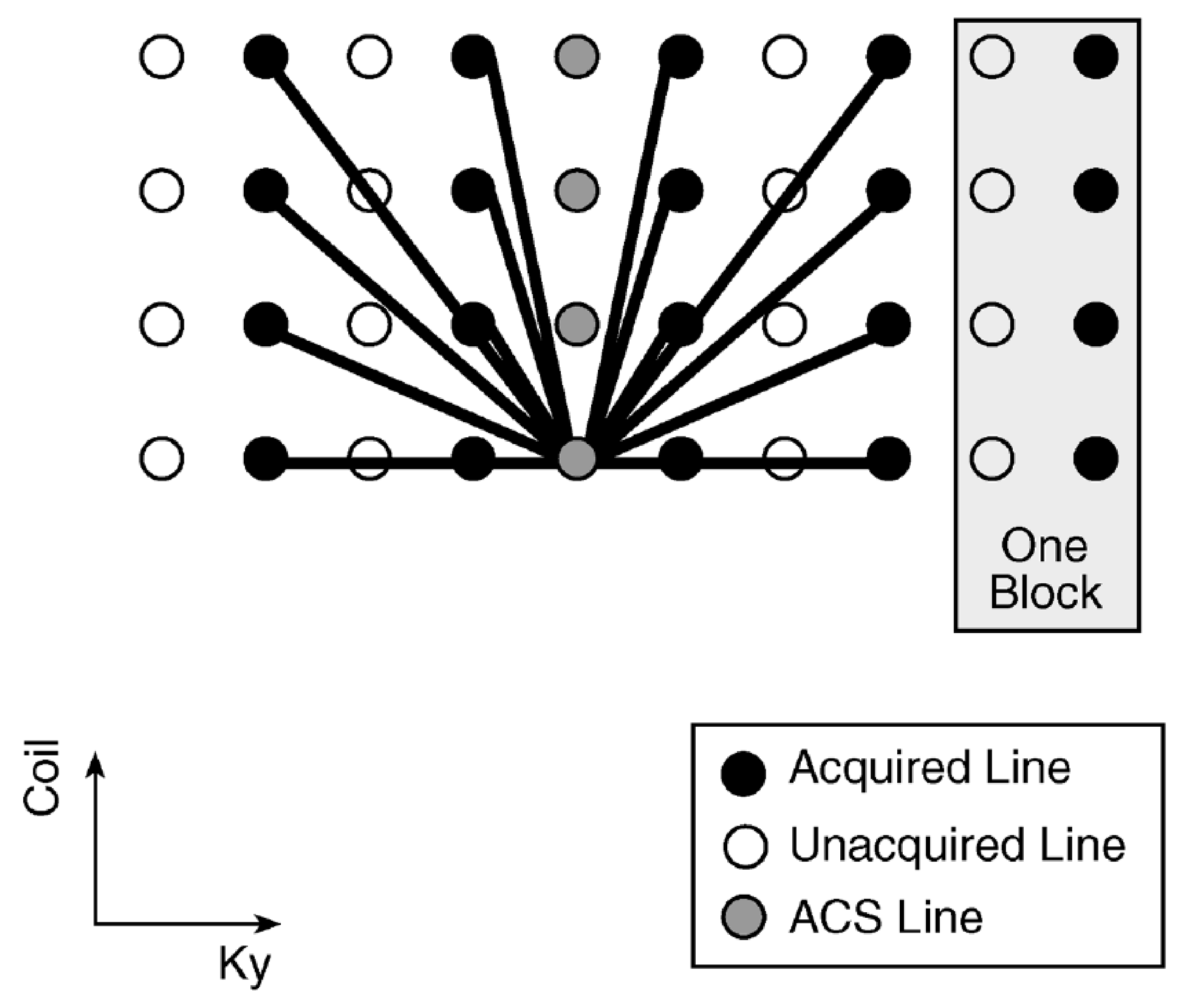
GRAPPA Reconstruction

- How do we find missing data from these samples?

$$\underset{\substack{\text{missing data}\\\text{for each coil}}}{\underline{\hat{m}_k(k_x, k_y)}} = \sum_{i,j,k} \underset{\text{weights}}{\underline{a_{i,j,k}}} \cdot \underset{\substack{\text{neighborhood data}\\\text{for each coil}}}{\underline{m_k(k_x + i\Delta k_x, k_y + j\Delta k_y)}}$$

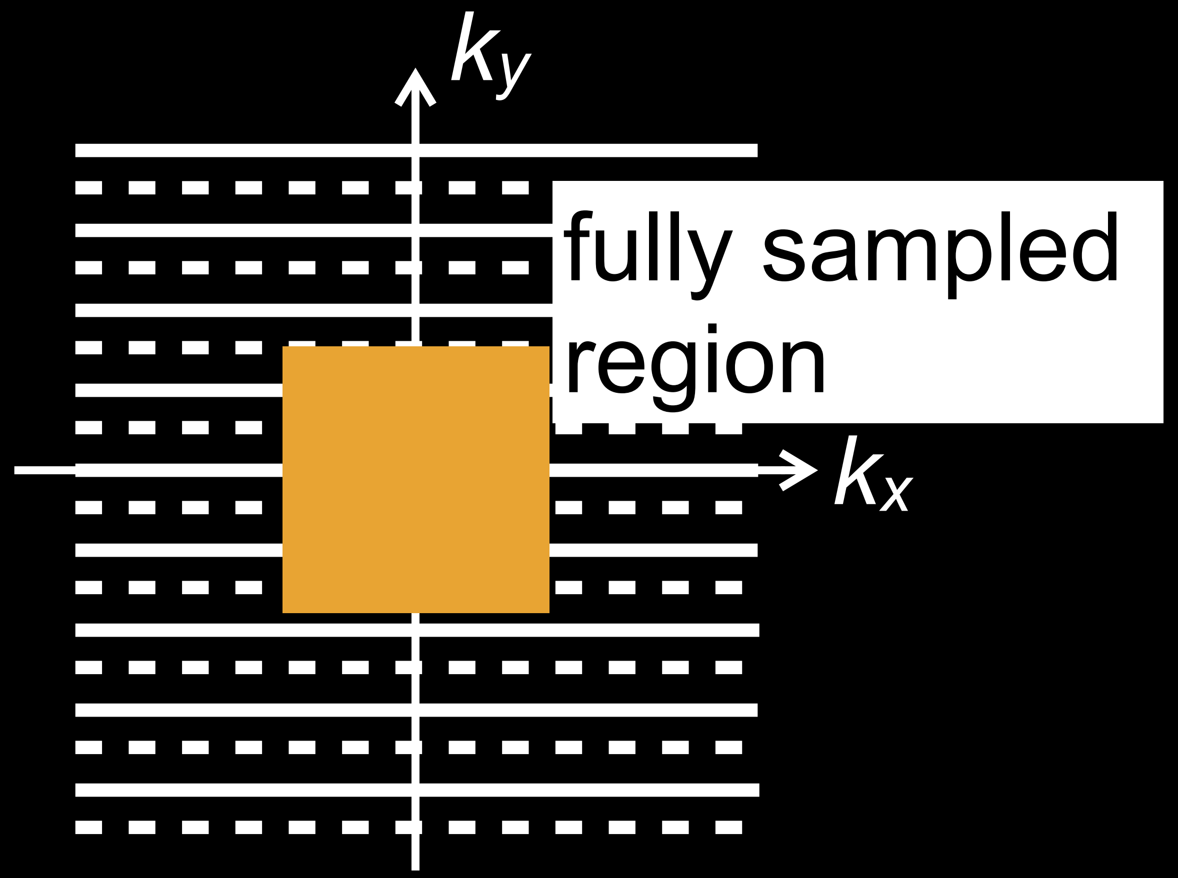
Auto-Calibration

$$\hat{m}_k(k_x, k_y) = \sum_{i,j,k} a_{i,j,k} \cdot m_k(k_x + i\Delta k_x, k_y + j\Delta k_y)$$



Auto-Calibration

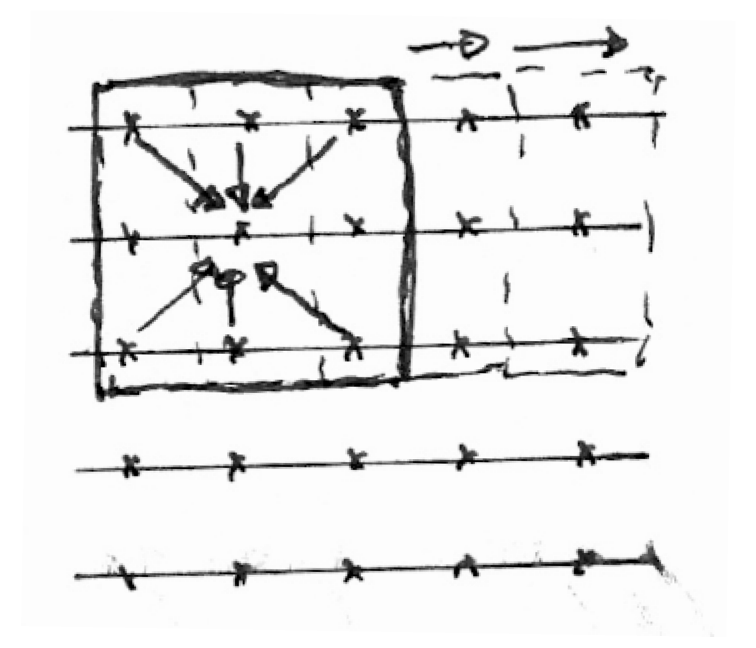
- Assume there is a fully sampled region
- We have samples of what the GRAPPA synthesis equations should produce



- Invert this to solve for GRAPPA weights

Auto-Calibration

- Calibration area has to be larger than the GRAPPA kernel
- Each shift of kernel gives another equation



- Here, 3x3 kernel, 5x5 calibration area gives 9 equations

Auto-Calibration

$$\hat{m}_k(k_x, k_y) = \sum_{i,j,k} a_{i,j,k} \cdot m_k(k_x + i\Delta k_x, k_y + j\Delta k_y)$$

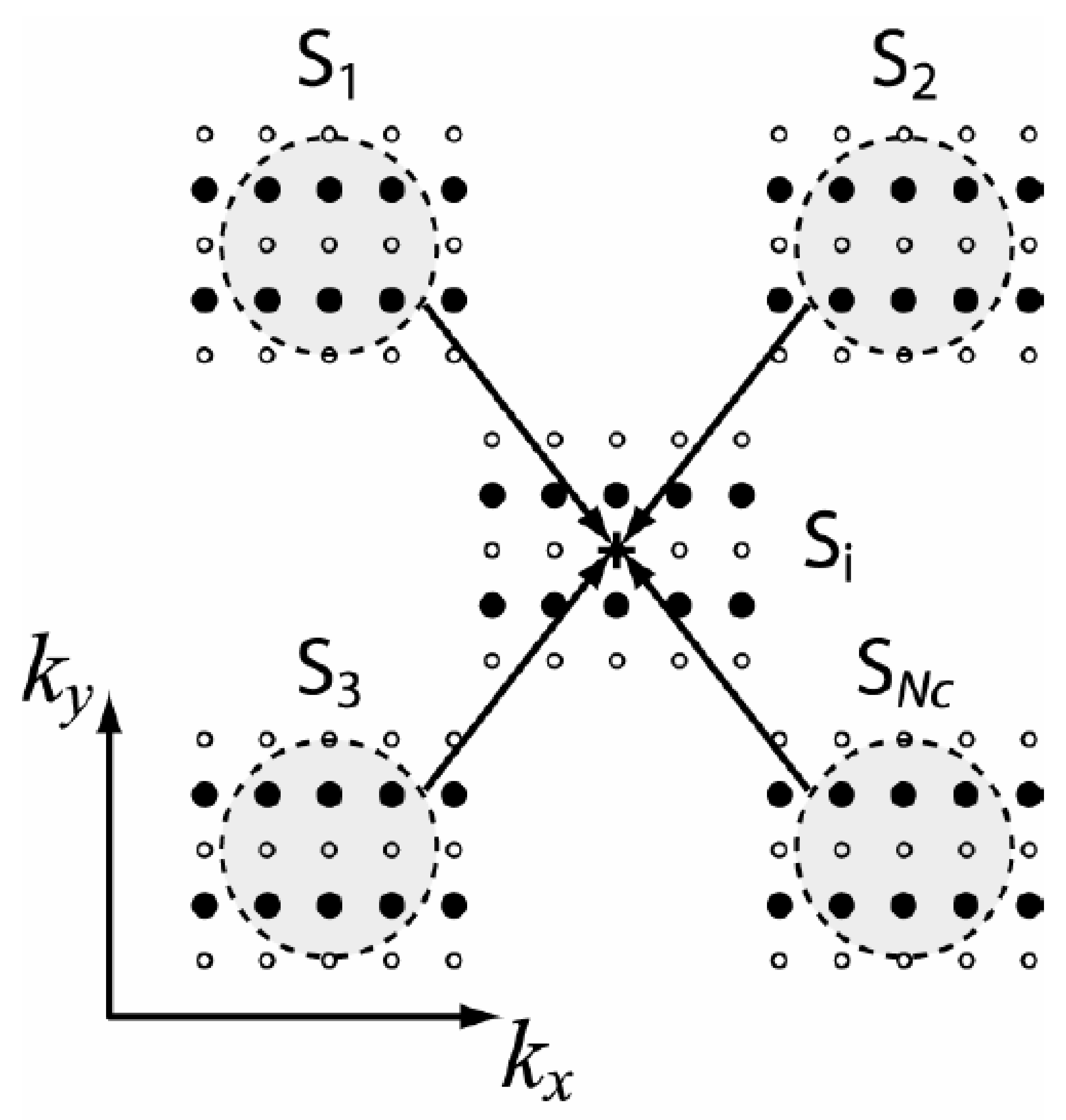
- Write as a matrix equation

$$\underset{\text{Calibration Data}}{\underline{M_{k,c}}} = \underset{\text{Neighborhood Data}}{\underline{M_A}} \cdot \overset{\text{GRAPPA Coefficients}}{\overline{a_k}}$$

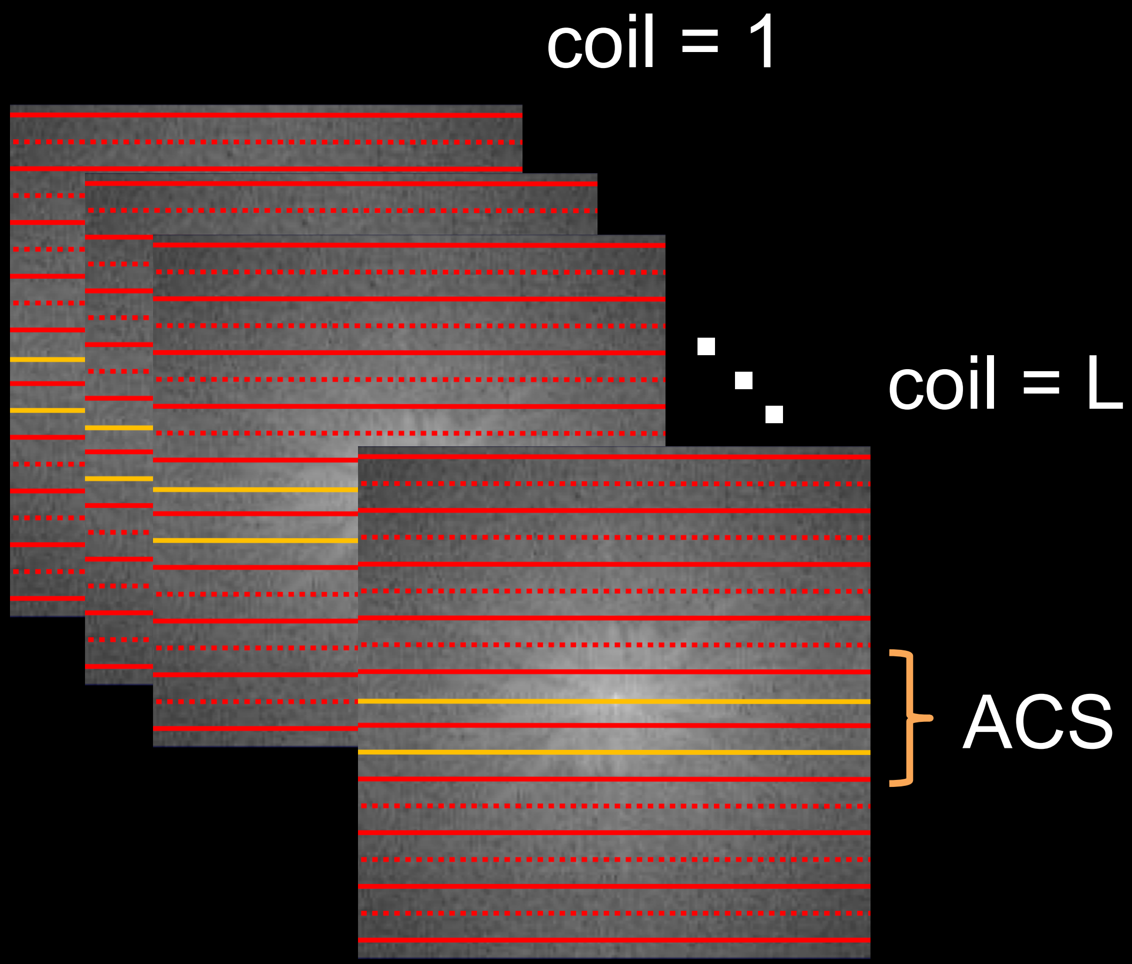
- GRAPPA weights are:

$$a_k = (M_A^* M_A + \lambda I)^{-1} M_A^* M_{k,c}$$

GRAPPA - Synthesis



Auto-Calibration Parallel Imaging



ACS (Auto-Calibration Signal) lines

$$\sum_{l=1}^{L} S_l^{ACS}(k_y - m\Delta k_y) = \sum_{l=1}^{L} n(l, m) S_l(k_y)$$

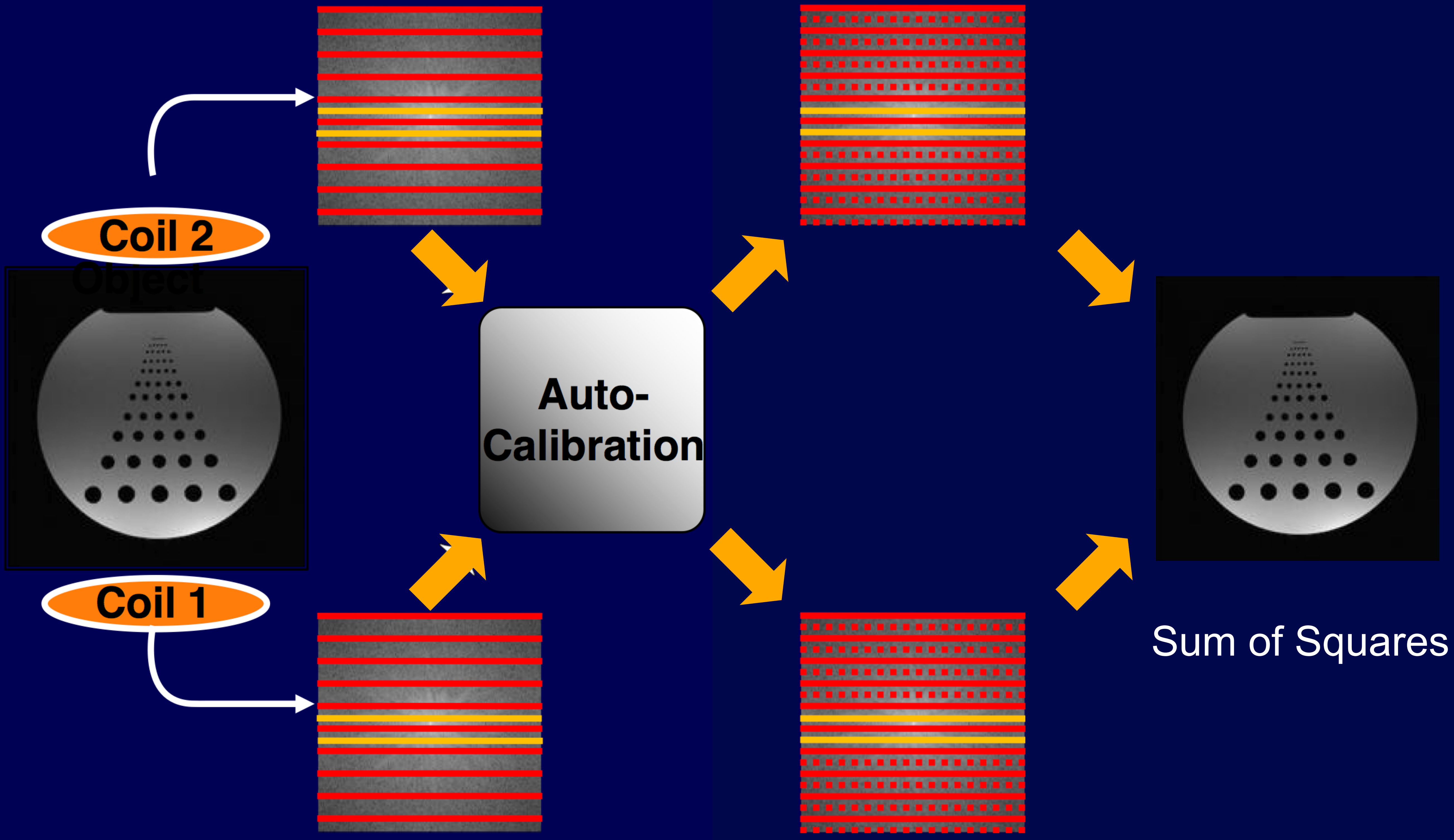
GRAPPA formula to reconstruct signal in one channel

$$S_j(k_y - m\Delta k_y) = \sum_{l=1}^{L} \sum_{b=0}^{N_b - 1} n(j, b, l, m) S_l(k_y - bA\Delta k_y)$$

A: Acceleration factor
n(j,b,l,m): GRAPPA weights

Griswold et al. MRM, 47(6):1202-1210 (2002)

GRAPPA Reconstruction



Coil 2

Object

Coil 1

Auto-Calibration

Sum of Squares

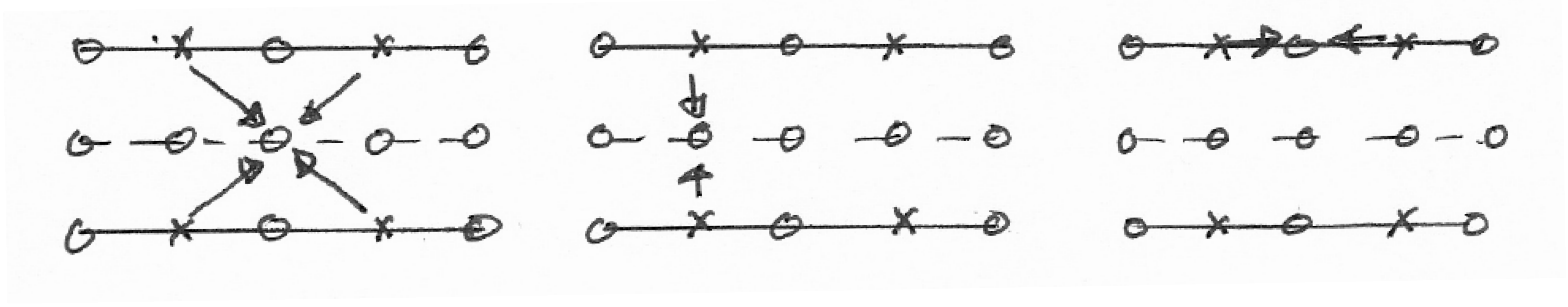
Reconstruct Missing k-space

GRAPPA

- Compute GRAPPA weights from calibration region
- Compute missing k-space data using the GRAPPA weights
- Reconstruct individual coil images
- Combine coil images

Considerations of GRAPPA

- Calibration region size
- GRAPPA kernel size
- Sample geometry dependence



GRAPPA

- Compute GRAPPA weights from calibration region
- Compute missing k-space data using the GRAPPA weights
- Reconstruct individual coil images
- Combine coil images

Summary

- Parallel imaging utilizes coil sensitivities to increase the speed of MRI
- Cases for parallel imaging
  - Higher patient throughput,
  - Real-time imaging/Interventional imaging
  - Motion suppression
- Cases against parallel imaging
  - SNR starving applications

Summary

- Many approaches:
  - Image domain - SENSE
  - k-space domain - SMASH, GRAPPA
  - Hybrid - ARC

- We will focus on two:
  - SENSE: optimal if you know coil sensitivities
  - GRAPPA: autocalibrating / robust

Further Reading

- Multi-coil Reconstruction
  - http://onlinelibrary.wiley.com/doi/10.1002/mrm.1910160203/abstract
- SENSE
  - http://www.ncbi.nlm.nih.gov/pubmed/10542355
- SMASH
  - http://www.ncbi.nlm.nih.gov/pubmed/9324327
- Parallel Imaging Overview
  - http://www.ncbi.nlm.nih.gov/pubmed/17374908

Thanks!

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https://mrrl.ucla.edu/sunglab/