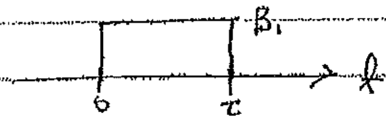
* Small tip-angle example

- consider a rectangular RF pulse (duration of τ)

$$B_1(t) = B_1 \cdot \Pi\left(\frac{t-\tau/2}{\tau}\right)$$

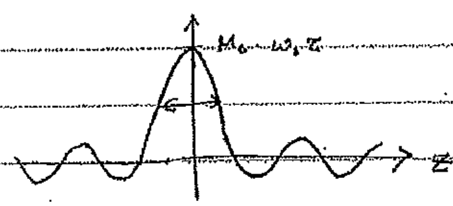


- $M_r(\tau, z) = i M_0 e^{-i\omega(z)\tau/2} \, \mathcal{F}_{1D}\left\{ \omega_1\left(t+\frac{\tau}{2}\right)\right\}\Big|_{f=\frac{-\gamma}{2\pi}G_z\cdot z}$

$$\omega_1\left(t+\frac{\tau}{2}\right) = \underbrace{\gamma\cdot B_1}_{\omega_1}\cdot \Pi\left(\frac{t}{\tau}\right)$$

$$\mathcal{F}_{1D}\left\{\Pi\left(\frac{t}{\tau}\right)\right\} = \tau \,\mathrm{sinc}(\tau\cdot f)$$

$$\Rightarrow M_r(\tau, z) = i\cdot M_0 \underbrace{e^{-i\omega(z)\tau/2}}_{\text{canceled out by the refocusing pulse}} \cdot \omega_1\tau\,\mathrm{sinc}\left(\tau\cdot\frac{\gamma}{2\pi}G_z\cdot z\right)$$



$$\Delta z = \frac{1}{\frac{\tau}{2\pi}\gamma G_z}$$

* Full analytical solution

$$\frac{d\vec{M}}{dt} = \begin{bmatrix} 0 & \omega(z) & 0 \\ -\omega(z) & 0 & \omega_1(t) \\ 0 & -\omega_1(t) & 0 \end{bmatrix} \vec{M}$$

$M_r = M_x + i M_y \leftarrow$ phasor representation

$$\frac{dM_r}{dt} = -i\,\omega(z)\,M_r + i\,\omega_1(t)\,M_z$$

$$\frac{dM_z}{dt} = -i\,\omega_1(t)\,M_y$$

Solution $\Longrightarrow$ very difficult $\Longrightarrow$ using approximation

$\sin\theta \simeq \theta$

$\cos\theta \simeq 1$

$M_z \simeq M_0 \simeq$ constant

$$\Rightarrow \frac{dM_z}{dt} = 0$$

$$\frac{dM_r}{dt} = -i\,\omega(z)\,M_r + i\,\omega_1(t)\,M_0$$

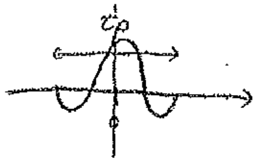
$$\frac{dy}{dx} + p(x)\,y = q(x)$$

$$y = \frac{\int u(x)\,q(x)\,dx}{u(x)}, \quad \text{where} \quad u(x) = \exp\left\{\int p(x)\,dx\right\}$$

$$\underbrace{\frac{d\overset{y}{M_r}}{dt}}_{x} + \underbrace{i\,\omega(z)}_{p(x)}\,\overset{y}{M_r} = \underbrace{i\,\omega_1(t)\,M_0}_{q(x)}$$

$$M_r(t,z) = i\,M_0\,e^{-i\omega(z)\cdot t}\int_0^t \omega_1(\tau)\,e^{-i\omega(z)\cdot\tau}\,d\tau$$

Assume, the RF pulse is symmetric and peaks at $t=0$ such that the pulse ends at $t = \tau_p/2$, τ_p is the length of the RF pulse $\longrightarrow$ let $\tau' = \tau - \tau_p/2$



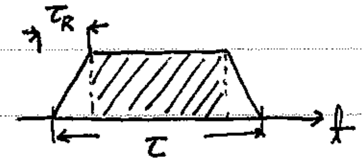
$$M_r(\tau_p) = i\,M_0\,e^{-\frac{i\omega(z)\tau_p}{2}}\int_{-\tau_p/2}^{\tau_p/2}\omega_1\left(t+\frac{\tau_p}{2}\right)e^{-i\omega(z)t}\,dt$$

assuming $\omega_1\left(t+\frac{\tau_p}{2}\right) = \begin{cases} 0 & \text{for } |t| > \frac{\tau_p}{2} \\ \omega_1\left(t+\frac{\tau_p}{2}\right) & \text{for } |t| < \frac{\tau_p}{2}\end{cases}$

$$\Rightarrow M_r(\tau,z) = i\,M_0\,e^{-i\omega(z)\tau/2}\,\mathcal{F}_{1D}\left\{\omega_1\left(t+\frac{\tau}{2}\right)\right\}\Big|_{f=-f(z) = -(\gamma/2\pi)G_z\cdot z}$$

$$* \quad |M_{xy}(\tau, z)| = M_0 \, \mathcal{F}_{1D} \left\{ W_1 \left(t + \frac{\tau}{2} \right) \right\} \Big|_{f = -\frac{\gamma}{2\pi} G_z z}$$

* Readout lobes



ex> $\tau = 1\,ms$, $\tau_R = 1/4\,ms$

$$2k_{x,max} = \frac{\gamma}{2\pi}(\tau - 2\tau_R)\,G_{max}$$

$$= 4.257\ KHz/G \cdot \frac{1}{2}\,ms \cdot 4\,G/cm$$

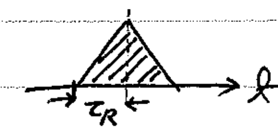
$$= 8.514\ cycles/cm$$

$$\Delta X = \frac{1}{2k_{x,max}} = \frac{1}{8.514\ cycles/cm} \cong 0.12\,cm \quad (TBW=1)$$

with a TBW = 4 pulse (Typical)

$$4 \cdot \Delta x \cong 0.47\ cm$$

* Blips



$$\Delta k_y = \frac{\gamma}{2\pi}\ \frac{1}{2} \cdot 2\tau_R\ G_{max}$$

$$= 4.257\ kHz/G \cdot \frac{1}{4}\,ms\ \ 4\,G/cm$$

$$= 4.257\ cycles/cm$$

Assume $L = 11$ (k-space lines)

$$2k_{y,max} = (L-1)\Delta k_y = 42\ cycles/cm$$

$$\Delta y = \frac{1}{2k_{y,max}} = 0.024\ cm$$

$$FOV = \frac{1}{\Delta k_y} = 0.23\ cm$$