Fundamental Math for MRI

M219 - Principles and Applications of MRI

Kyung Sung, Ph.D.

1/26/2022

Course Overview

- Course website
  - https://mrrl.ucla.edu/pages/m219
- Course schedule
  - https://mrrl.ucla.edu/pages/m219_2022
- Assignments
  - Homework #1 due on 1/26 by 5pm
  - Homework #2 will be out on 1/26

Course Overview

- Office Hours
  - TA (Ran Yan) - Tuesday 4-5pm
    https://uclahs.zoom.us/j/96870184581?pwd=VkczL0lyRkxsQ3FHcnIxQ1M2U3hPdz09

    Password: 900645

  - Instructor (Kyung Sung) - Friday 2-3pm
    https://uclahs.zoom.us/j/94058312815?pwd=TkI3ajhkamdGTnhqOVNnbk5RMnJGQT09

    Password: 888767

#8	1/26 Wed	Fundamental Math of MRI	Homework #1 due, Homework #2 out • 2D Fourier transform • Fourier transform and its applications
#9	1/31 Mon	Spatial Localization I	
#10	2/2 Wed	Spatial Localization II	
#11	2/7 Mon	MRI Signal Equation and Basic Image Reconstruction (by Dr. Holden Wu)	
#12	2/9 Wed	Fast Imaging and Advanced Image Reconstruction (by Dr. Holden Wu)	
#13	2/14 Mon	Basics of MR Spectroscopy (by Dr. Albert Thomas)	Homework #2 due, Homework #3 out

Last Time …

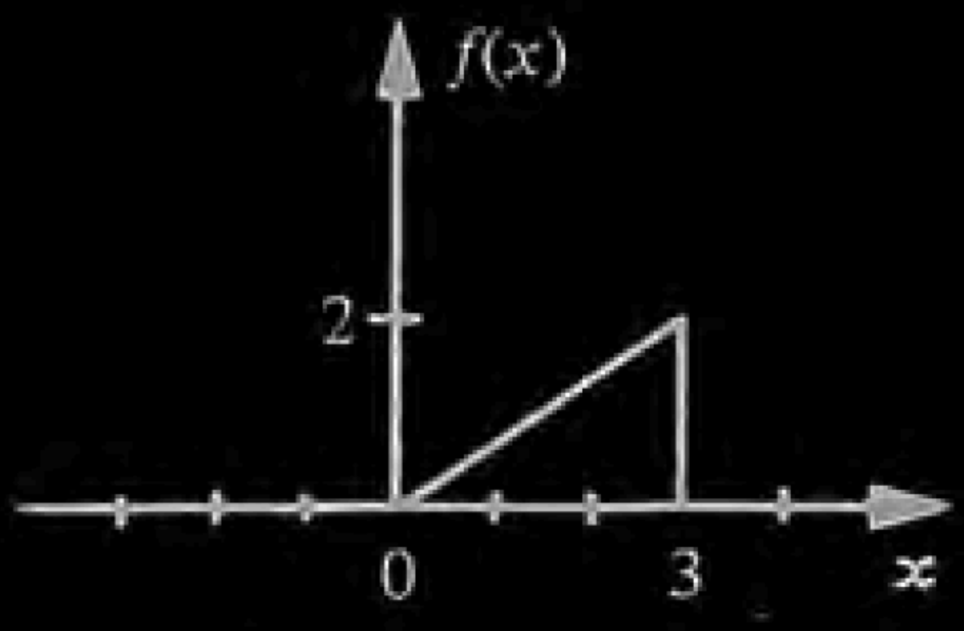
Convolution

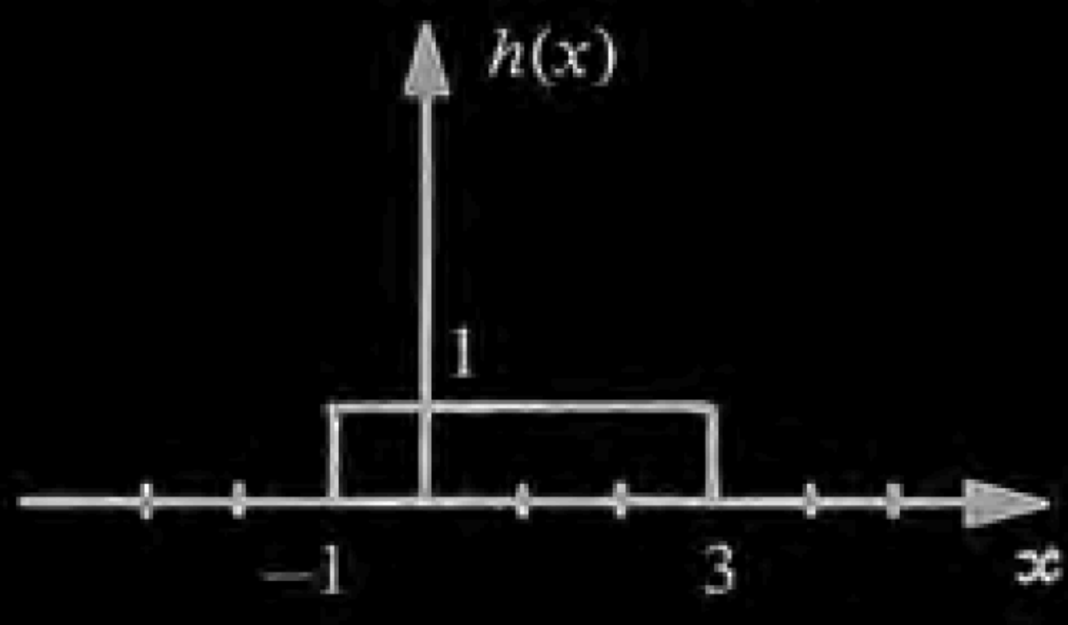
$$f(x) = \int_{-\infty}^{\infty} g(\tau)h(x-\tau)d\tau = \int_{-\infty}^{\infty} g(x-\tau)h(\tau)d\tau$$

$$f(x) = g(x) * h(x)$$

$$f[n] = g[n] * h[n] = \sum_{m=-\infty}^{\infty} g[m]h[n-m]$$

Convolution-Graphical Illustration



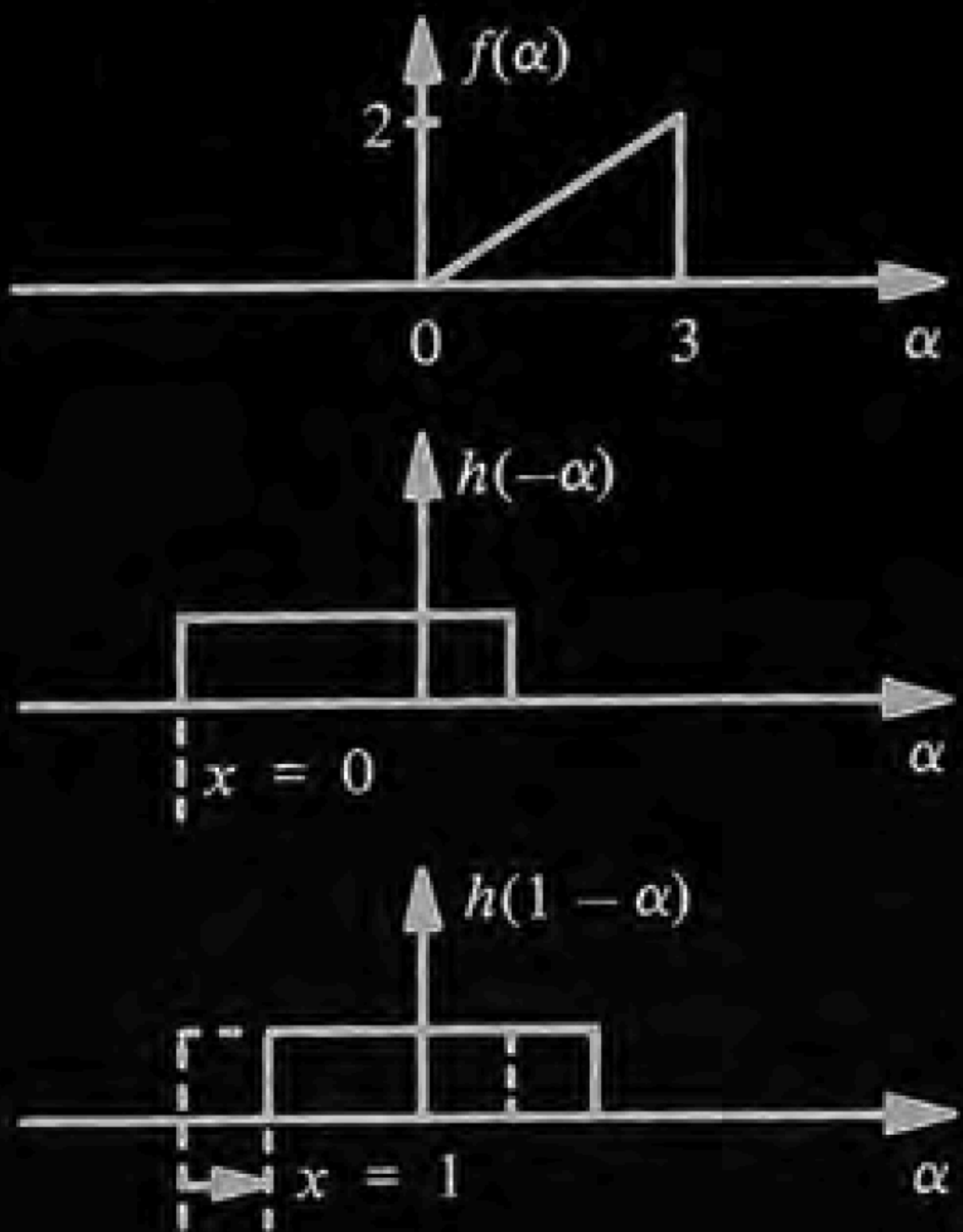


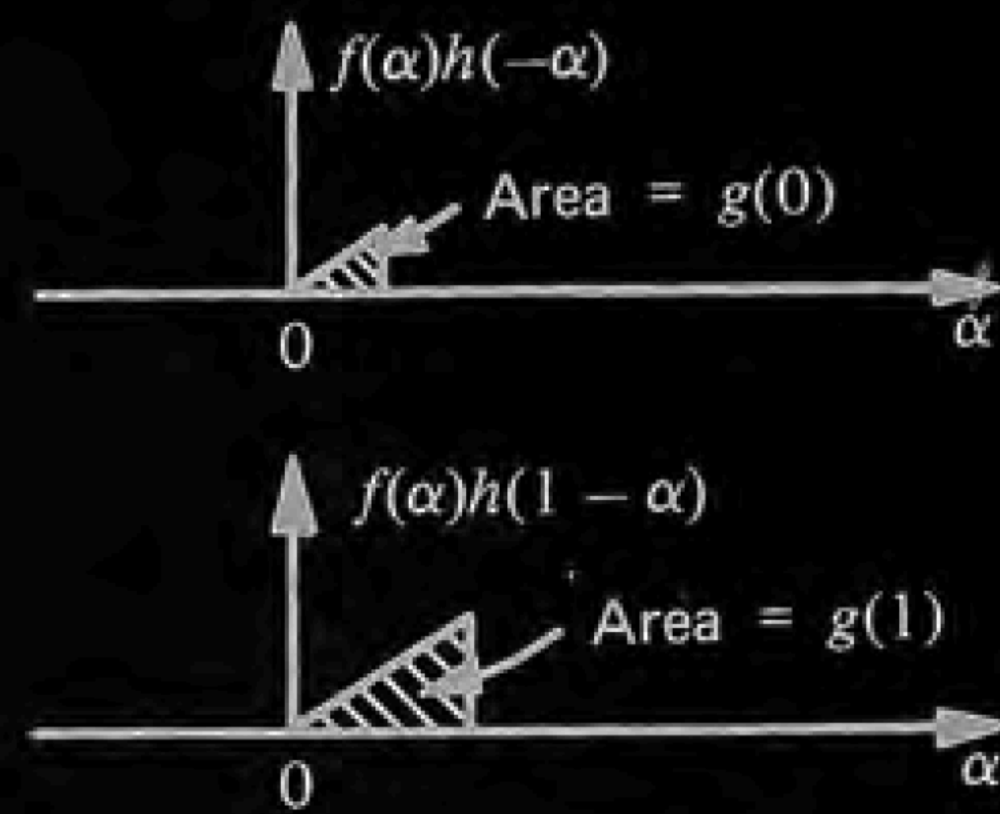
How to do f(x)*h(x)

$$g(x) = f(x) * h(x)$$

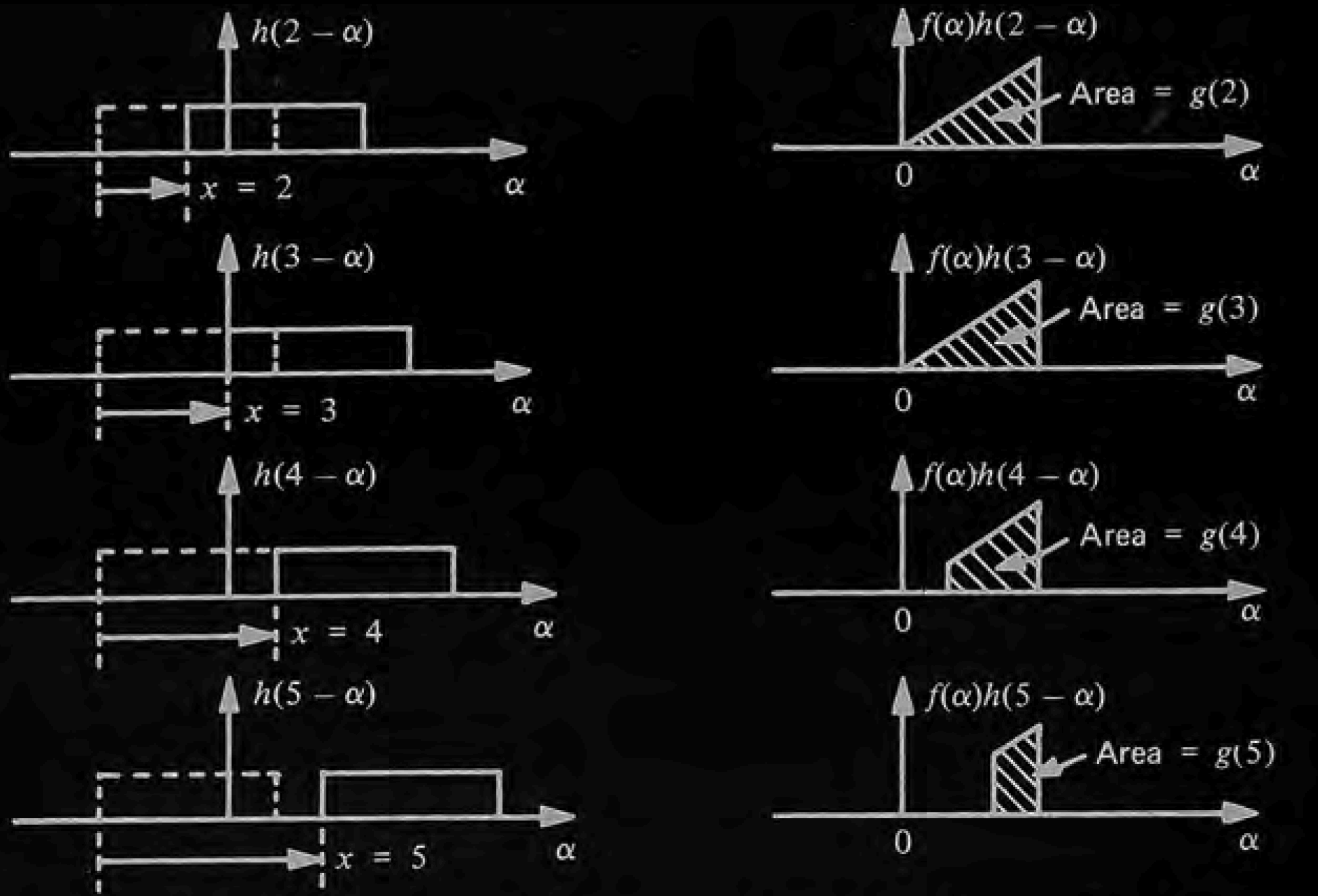
Convolution-Graphical Illustration

$$g(x) = \int_{-\infty}^{\infty} f(\alpha)h(x-\alpha)d\alpha$$

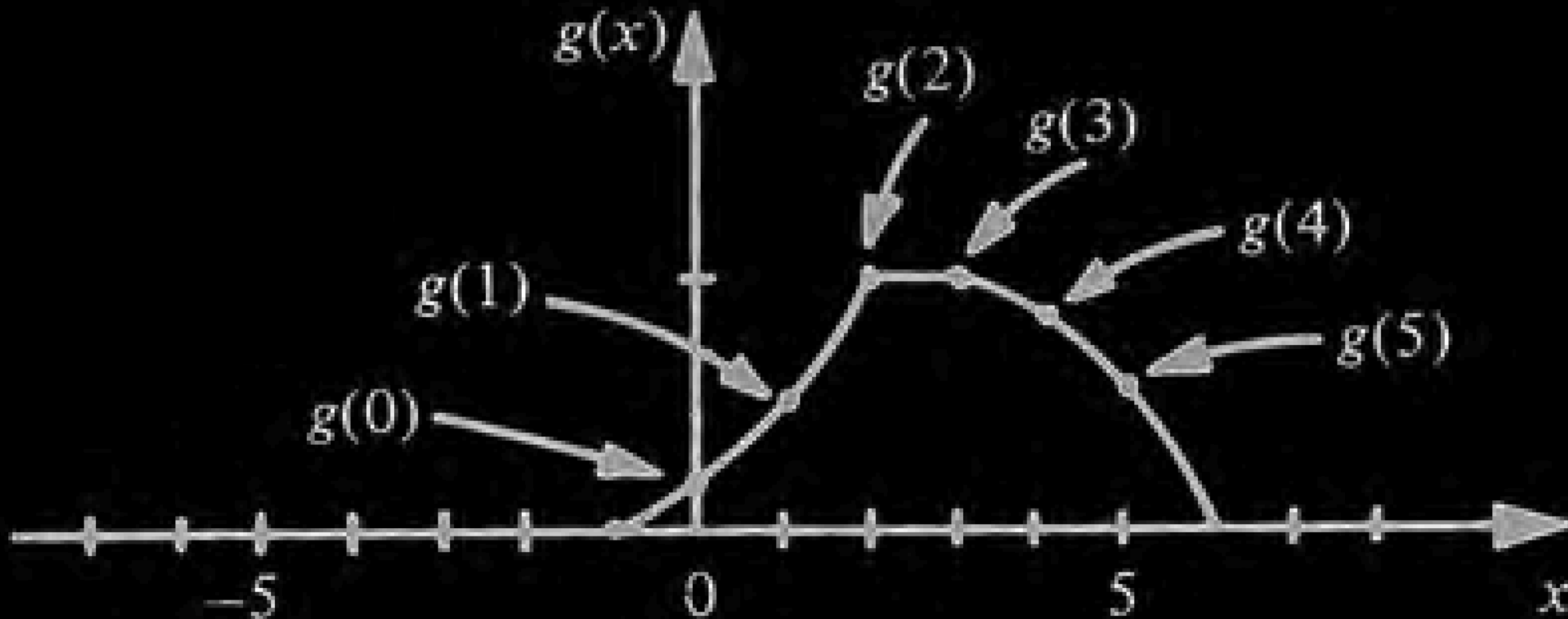




Convolution-Graphical Illustration



Convolution-Graphical Illustration



$$g(x) = f(x) * h(x)$$

Properties of Convolution

Commutativity:

$$g * h = h * g$$

Associativity:

$$f * (g * h) = (f * g) * h$$

Distributivity:

$$f * (g + h) = f * g + f * h$$

Shifting property:

If $g * h = f$, then

$$g(x - x_0) * h(x) = g(x) * h(x - x_0) = f(x - x_0)$$

Convolution with Delta Function

$$f(x) * \delta(x) = \int_{-\infty}^{\infty} f(\alpha)\delta(x-\alpha)d\alpha$$

$$= \int_{-\infty}^{\infty} f(\alpha)\delta(a-x)d\alpha$$

$$= \int_{-\infty}^{\infty} f(x)\delta(a-x)d\alpha$$

$$= f(x)\int_{-\infty}^{\infty} \delta(x-\alpha)d\alpha$$

$$= f(x)$$

Convolution with Comb Function

$$f(x) * \text{III}(x)$$

$$= f(x) * \sum_{n=-\infty}^{\infty} \delta(x-n)$$

$$= \int_{-\infty}^{\infty} f(\tau) \sum_{n=-\infty}^{\infty} \delta(x-\tau-n)\, d\tau$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) \cdot \delta(x-\tau-n)\, d\tau$$

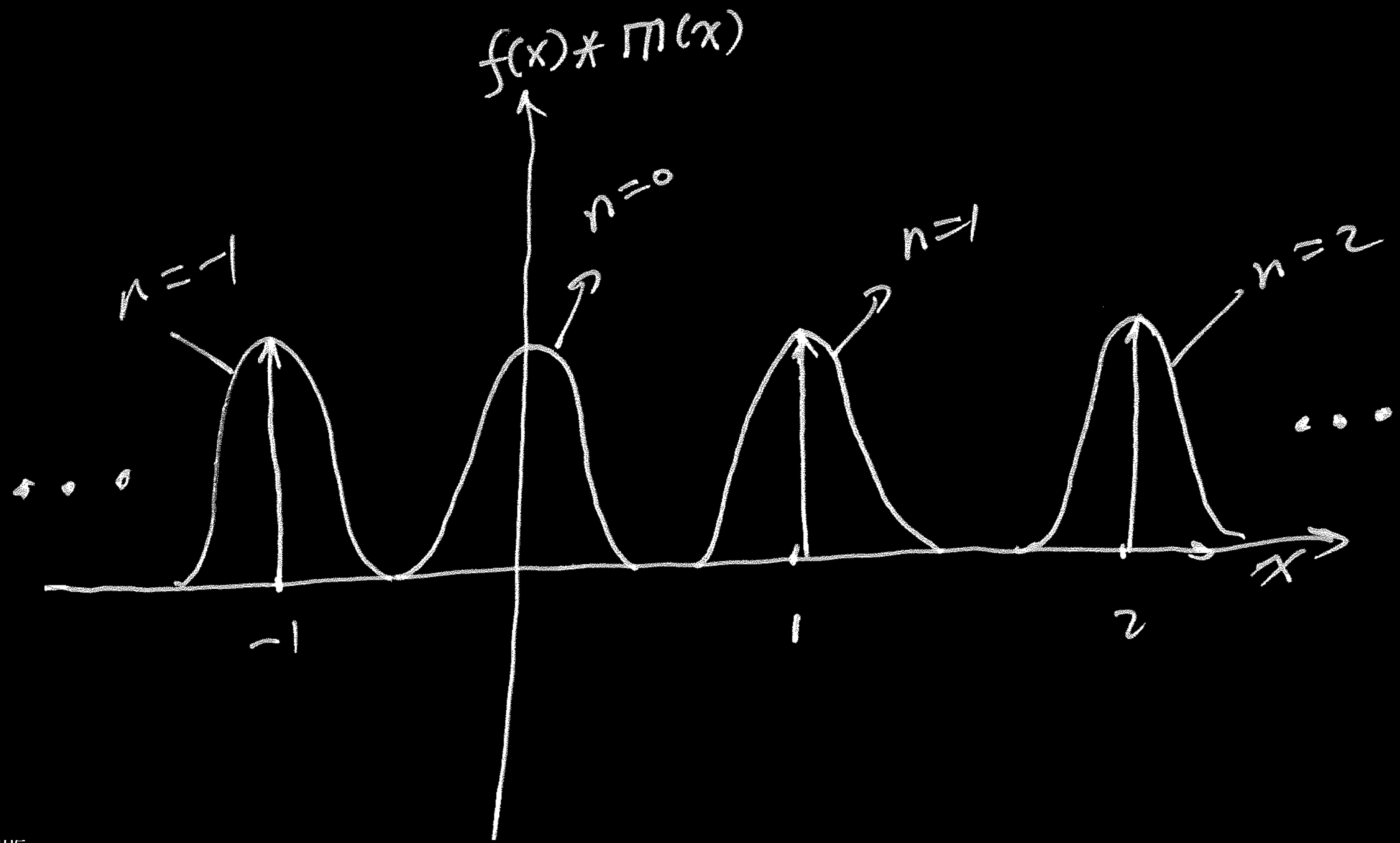
Convolution with Comb Function

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)\,\delta(\tau - x + n)\,d\tau$$

$$= \sum_{n=-\infty}^{\infty} f(x-n)$$

Convolution with Comb Function

$$f(x) * \text{III}(x) = \sum_{n=-\infty}^{\infty} f(x-n)$$



Fourier Transform

$$S(k) = \mathcal{F}\{\rho(x)\} = \mathcal{F}\rho = \int_{-\infty}^{\infty} \rho(x) e^{-i2\pi kx} dx$$

Inverse Fourier transform

$$\rho(x) = \mathcal{F}^{-1}\{S(k)\} = \int_{-\infty}^{\infty} S(k) e^{i2\pi kx} dk$$

Fourier Transform

Properties of FT

Uniqueness:

$$\rho_1(x) = \rho_2(x) \longrightarrow S_1(k) = S_2(k)$$

Linearity:

$$a\rho_1(x) + b\rho_2(x) \longleftrightarrow aS_1(k) + bS_2(k)$$

Shifting theorem:

Important!

$$\rho(x - x_0) \longleftrightarrow S(k)e^{-i2\pi k x_0}$$

$$e^{i2\pi k_0 x}\rho(x) \longleftrightarrow S(k - k_0)$$

Properties of FT

Scaling property:

$$\rho(ax) \longleftrightarrow \frac{1}{|a|} S\left(\frac{k}{a}\right)$$

Conjugate symmetry:

$$\rho^*(x) \longleftrightarrow S^*(-k)$$

Convolution theorem:

$$\rho_1(x) * \rho_2(x) \longleftrightarrow S_1(k) S_2(k)$$

$$\rho_1(x)\rho_2(x) \longleftrightarrow S_1(k) * S_2(k)$$

Fourier Transform of Rect Function

$$f(x) = \text{rect}(x)$$

$$F(k) = \int_{-\infty}^{\infty} \text{rect}(\alpha) \cdot e^{-i2\pi k\alpha} \, d\alpha$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i2\pi k\alpha} \, d\alpha$$

$$= \frac{1}{-i2\pi k} e^{-i2\pi k\alpha} \Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{-i2\pi k} \left[e^{-i\pi k} - e^{i\pi k} \right]$$

$$= \frac{1}{-i2\pi k} \cdot -i \cdot 2 \cdot \sin \pi k$$

$$= \frac{\sin \pi k}{\pi k} = \text{Sinc}(k)$$

$$\text{rect}(x) \overset{\mathcal{F}}{\longleftrightarrow} \text{Sinc}(k)$$

Fourier Transform of Delta Function

$$\int_{-\infty}^{\infty} \varphi(x)\delta(x - x_0)dx = \varphi(x_0)$$

Delta Function Property

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(\alpha)e^{-j2\pi\xi\alpha}d\alpha$$

$$= e^{-j2\pi\xi\alpha}\Big|_{\alpha=0}$$

$$= 1.$$

$$\delta(x - x_0) \xrightarrow{\mathcal{F}} e^{-j2\pi x_0 \xi}$$

What is the FT of $e^{-i2\pi k_0 x}$ **?**

Fourier Transform of Delta Function

What is the FT of $e^{-i2\pi k_0 x}$ **?**

$$\int_{-\infty}^{\infty} e^{-i2\pi k_0 \cdot \alpha} e^{-i2\pi k\alpha} d\alpha$$

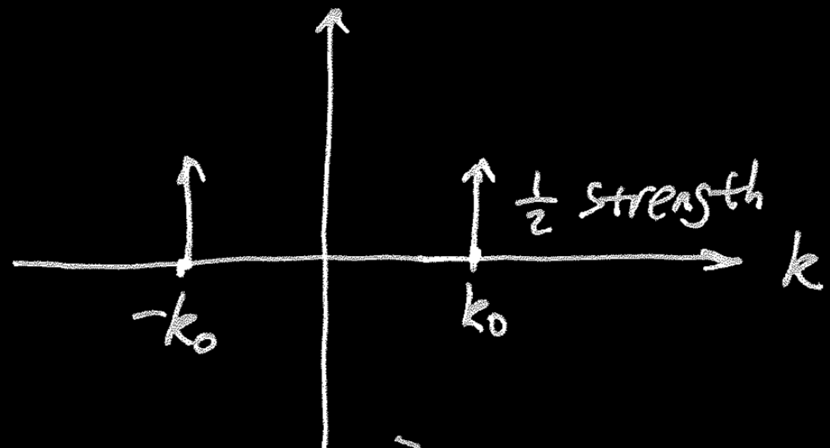
$$= \int_{-\infty}^{\infty} e^{-i2\pi(k+k_0)\alpha} d\alpha$$

$$= \delta(k+k_0)$$

FT of Sinusoidal Function

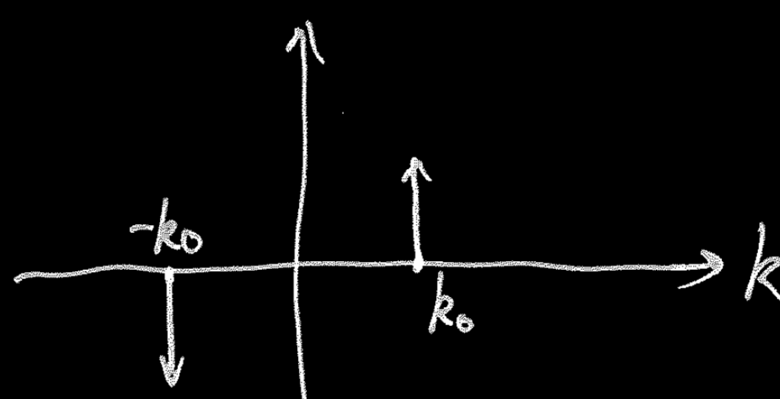
$$\cos(2\pi k_0 x) = \frac{1}{2}\left[e^{-i2\pi k_0 x} + e^{i2\pi k_0 x} \right]$$

$$\mathcal{F}[\cos(2\pi k_0 x)] = \frac{1}{2}\left[\delta(k-k_0) + \delta(k+k_0) \right]$$



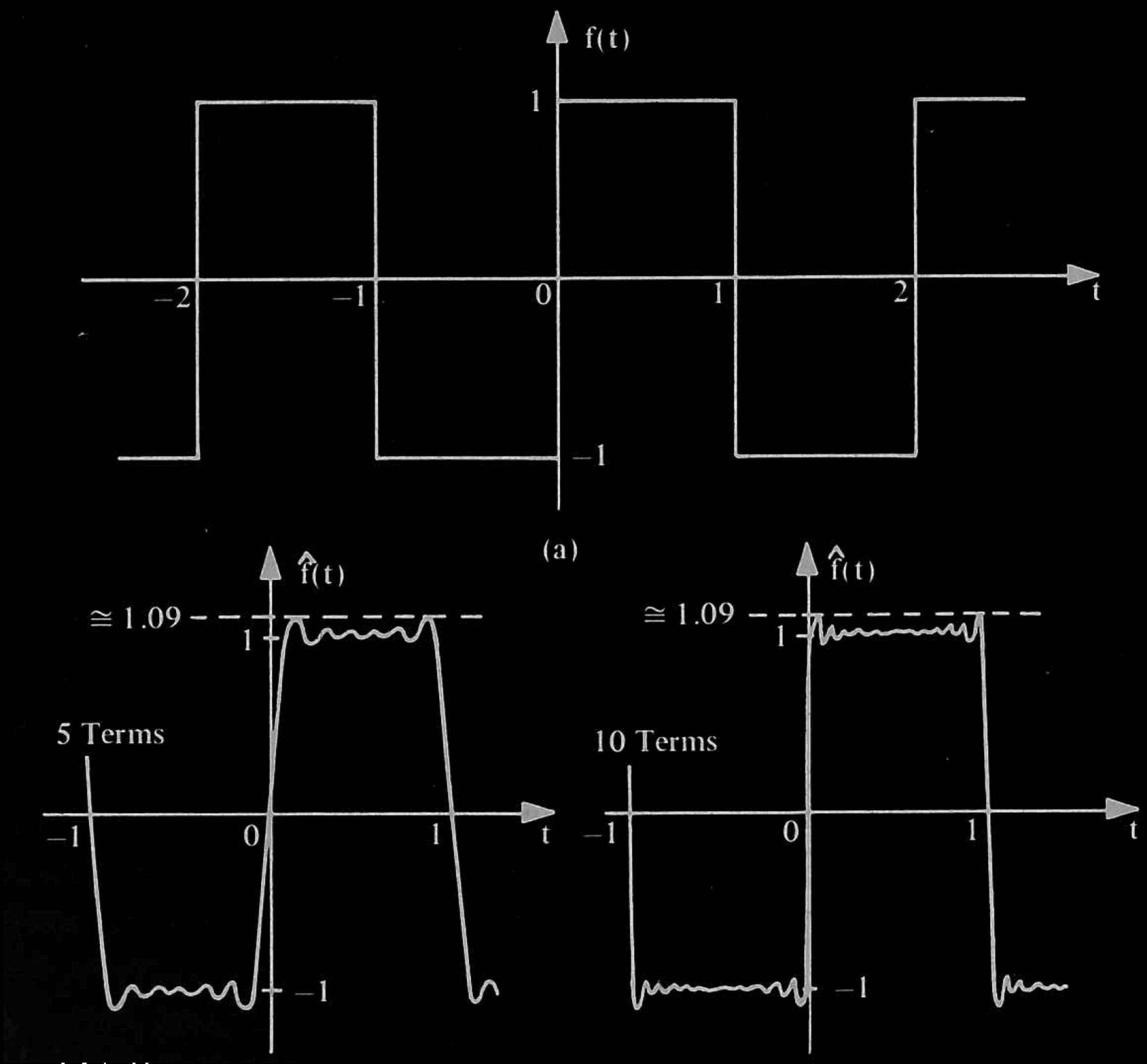
$$\mathcal{F}[\sin(2\pi k_0 x)] = \mathcal{F}\left\{ \frac{1}{2i}\left[e^{i2\pi k_0 x} - e^{-i2\pi k_0 x} \right] \right\}$$

$$= \frac{1}{2i}\left[\delta(k-k_0) - \delta(k+k_0) \right]$$



FT of Comb Function

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nx}$$



Fourier Series of Periodic Functions

Example of periodic rectangular function

FT of Comb Function

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nx}$$

$$c_n = \int_{-1/2}^{1/2} f(\alpha) e^{-j2\pi n\alpha} d\alpha$$

$$= \int_{-1/2}^{1/2} \mathrm{comb}(\alpha) e^{-j2\pi n\alpha} d\alpha$$

$$= \int_{-\infty}^{\infty} \delta(\alpha) e^{-j2\pi n\alpha} d\alpha$$

$$= 1,$$

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} e^{j2\pi nx}$$

FT of Comb Function

$$\mathcal{F}\{\mathrm{comb}(x)\} = \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} e^{j2\pi nx} \right\}$$

$$\mathcal{F}\{\mathrm{comb}(x)\} = \sum_{n=-\infty}^{\infty} \mathcal{F}\{ e^{j2\pi nx} \}$$

$$= \sum_{n=-\infty}^{\infty} \delta(\xi - n)$$

$$= \mathrm{comb}(\xi).$$

FT of Comb Function

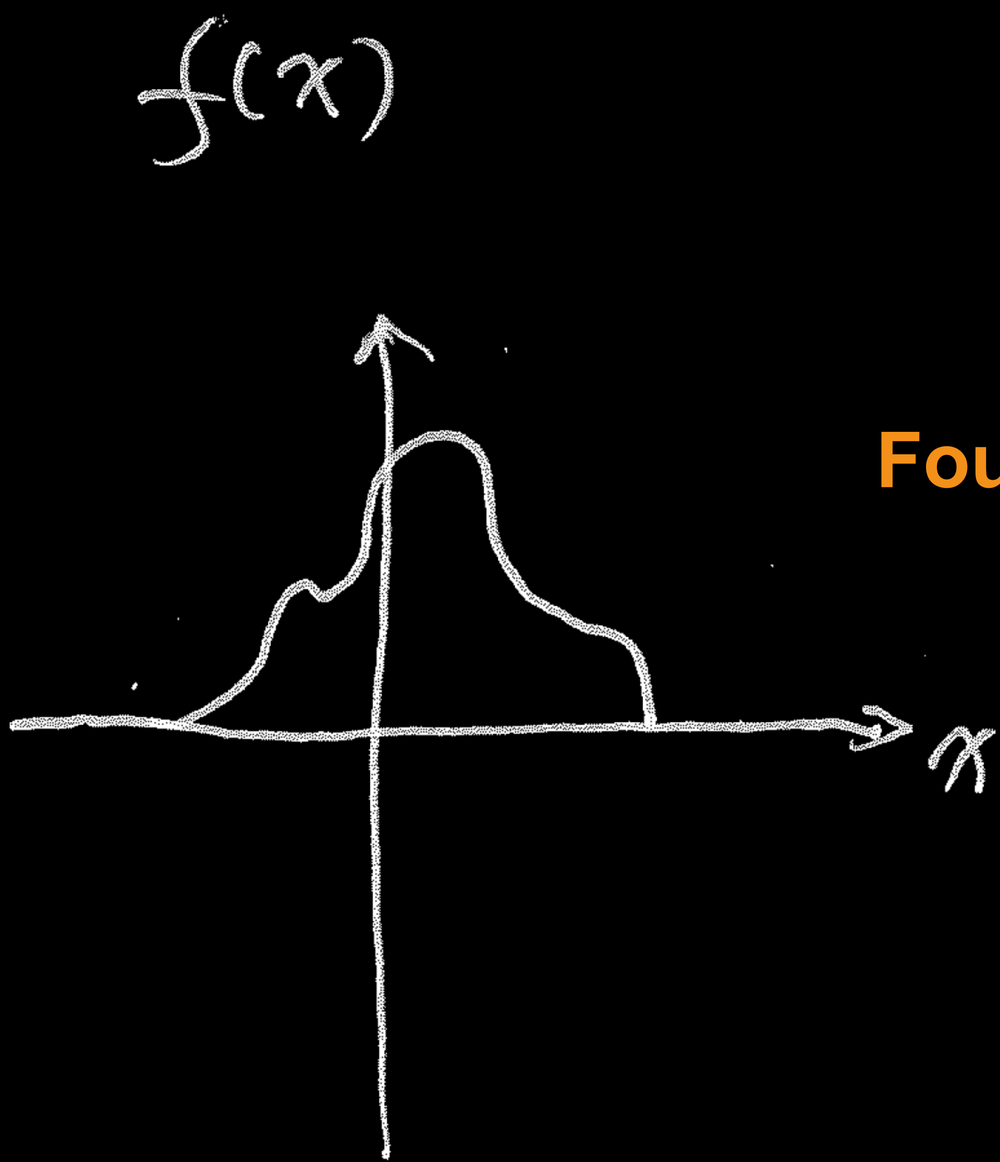
$$\sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) = \frac{1}{\Delta x} \mathrm{comb}\left(\frac{x}{\Delta x}\right)$$

$$\mathcal{F}\left[\frac{1}{\Delta x} \mathrm{comb}\left(\frac{x}{\Delta x}\right)\right] = \frac{1}{\Delta x} \cdot \Delta x \cdot \mathrm{comb}(\Delta x \cdot k)$$

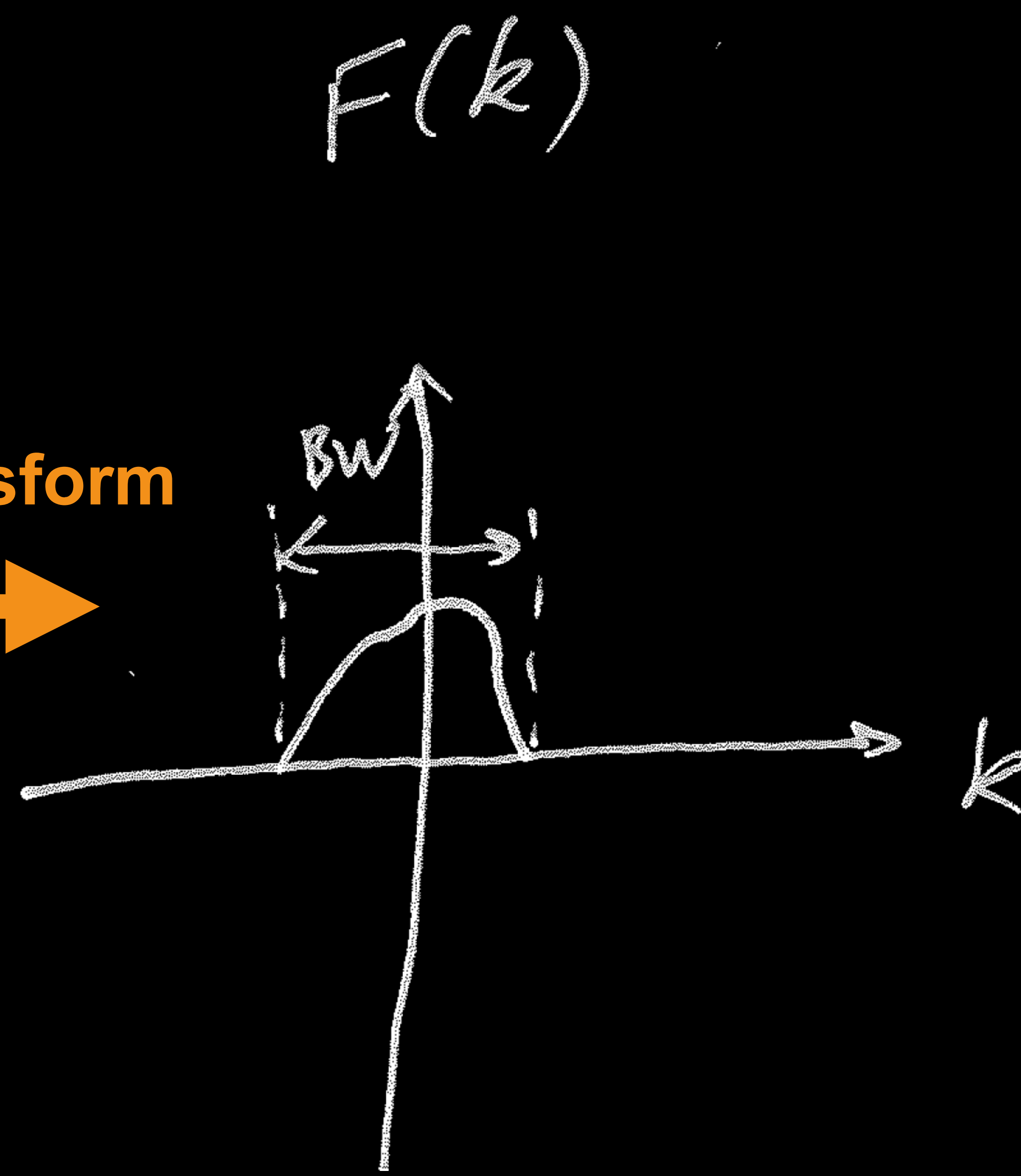
$$\Delta k \triangleq \frac{1}{\Delta x} \qquad = \Delta k \cdot \frac{1}{\Delta k} \mathrm{comb}\left(\frac{k}{\Delta k}\right)$$

$$= \Delta k \cdot \sum_{n=-\infty}^{\infty} \delta(k - n\Delta k)$$

Sampling Theory



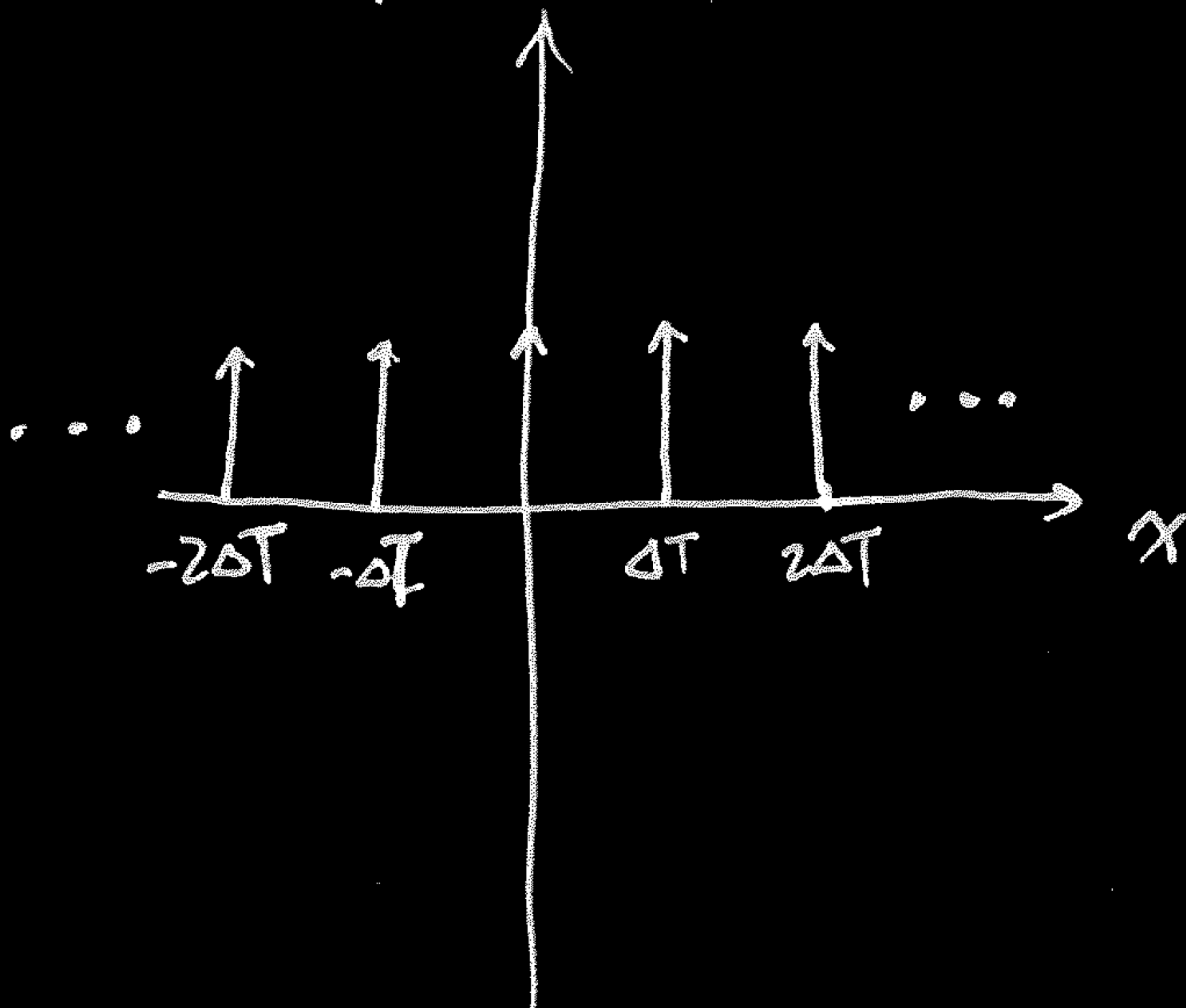
Fourier Transform



Limited Bandwidth (BW)

Sampling Function

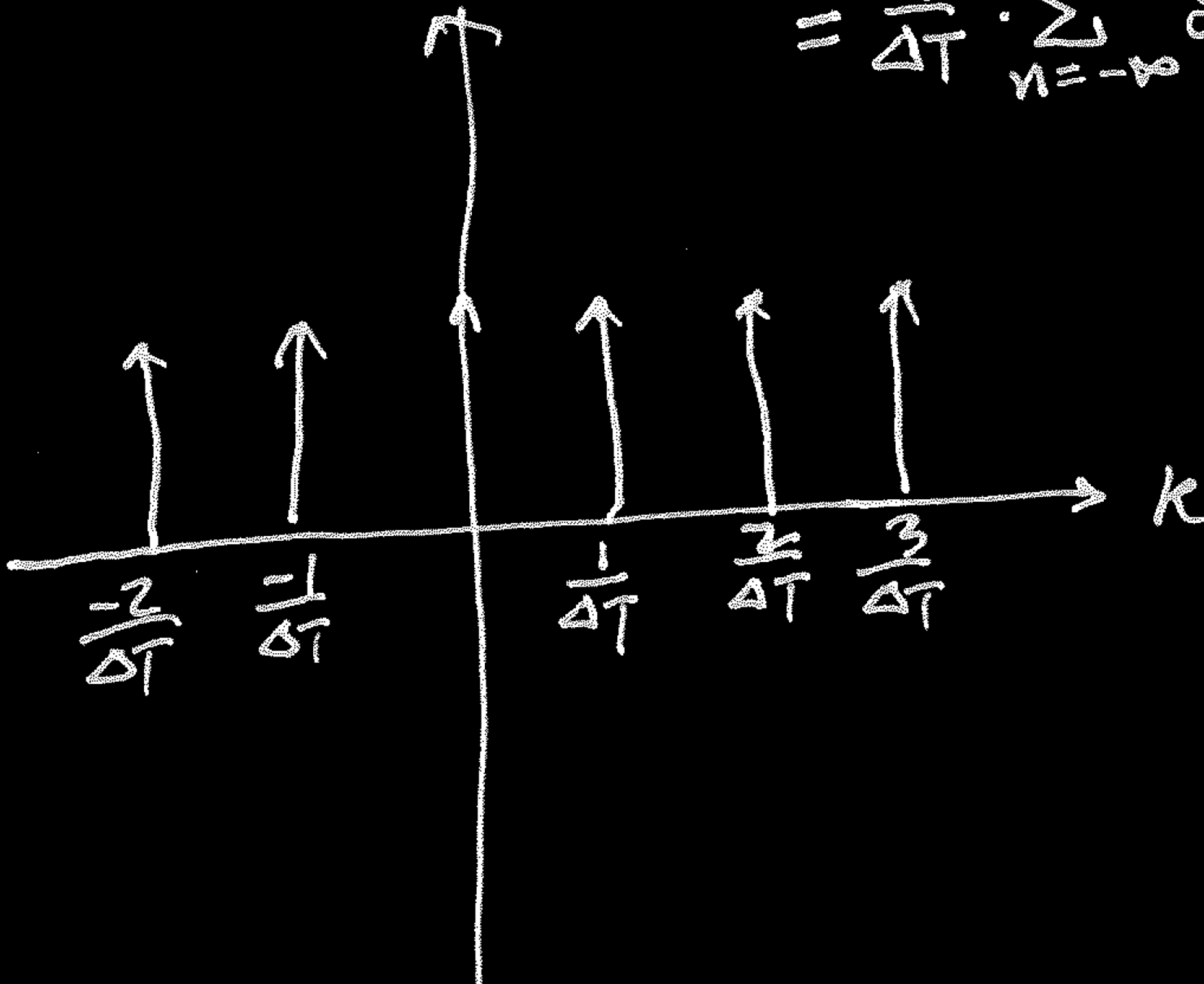
$$s(x) = \frac{1}{\Delta T}\, Ш\left(\frac{x}{\Delta T}\right)$$



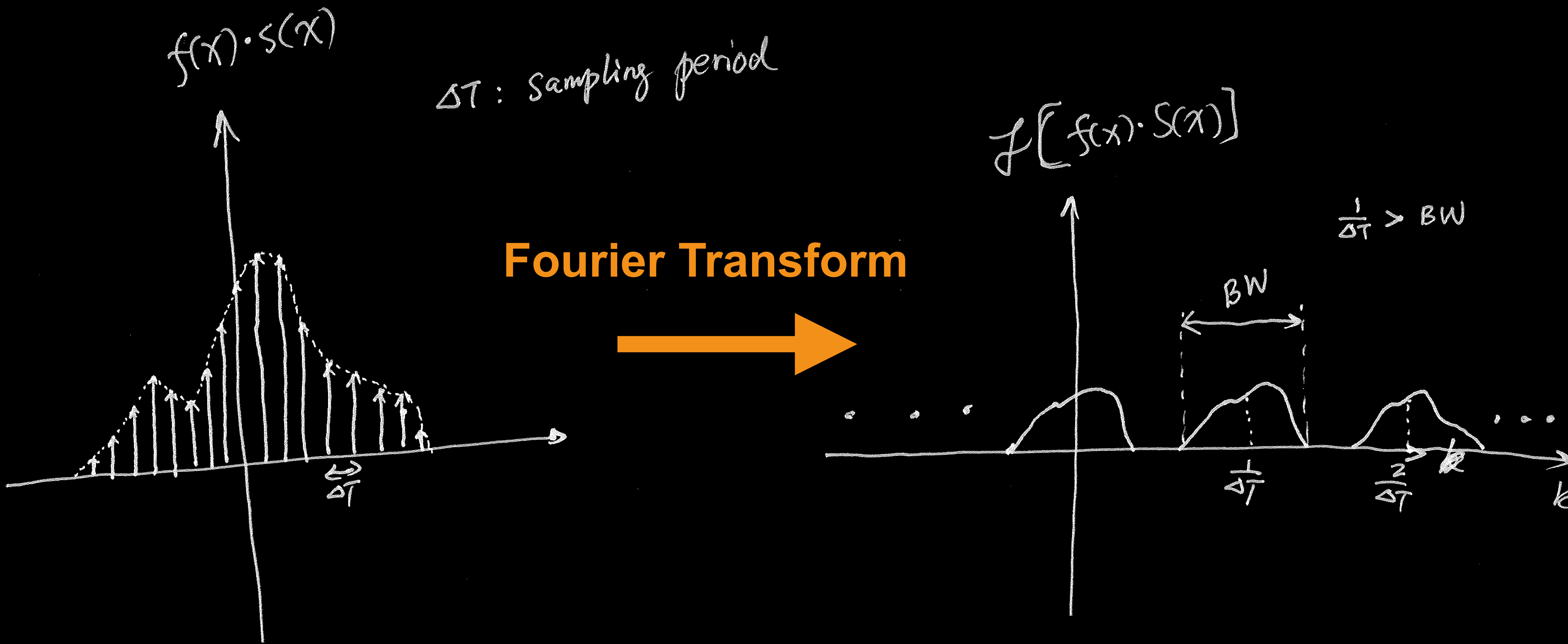
Sampling Function

$$\mathcal{F}[s(x)] = \text{III}(k \cdot \Delta T)$$

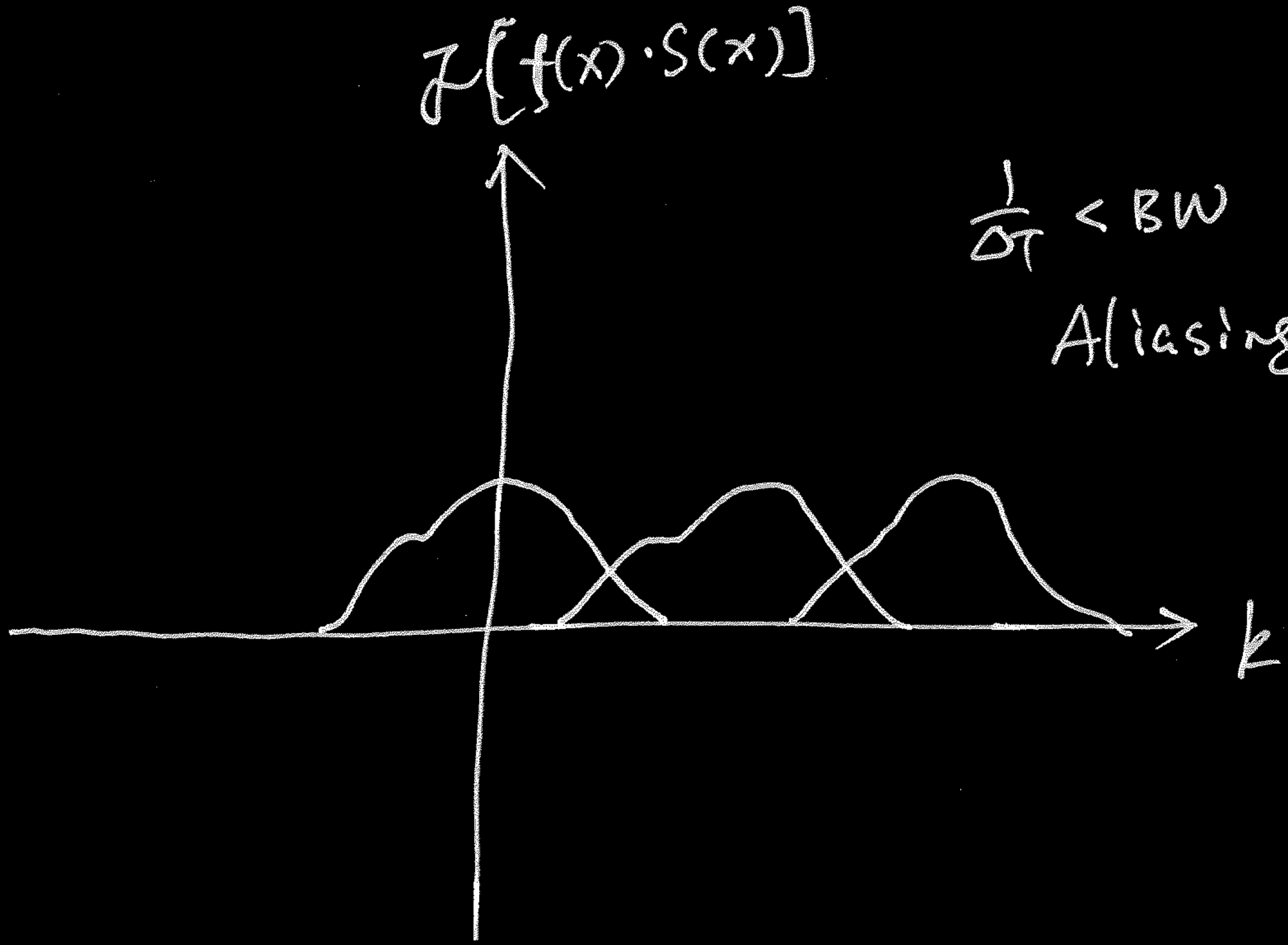
$$= \frac{1}{\Delta T} \cdot \sum_{n=-\infty}^{\infty} \delta\left(k - \frac{n}{\Delta T}\right)$$



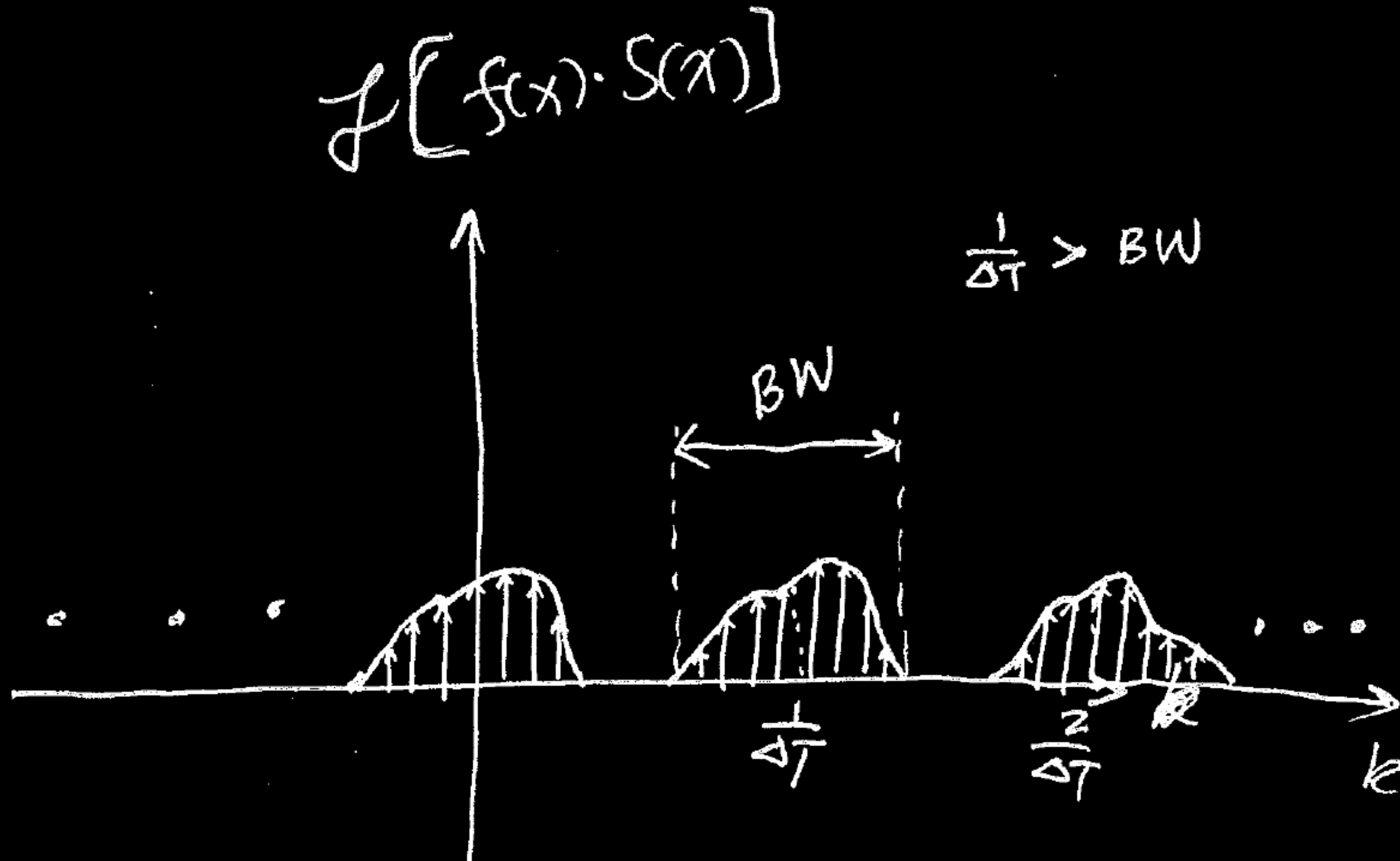
Sampling Function



Aliasing

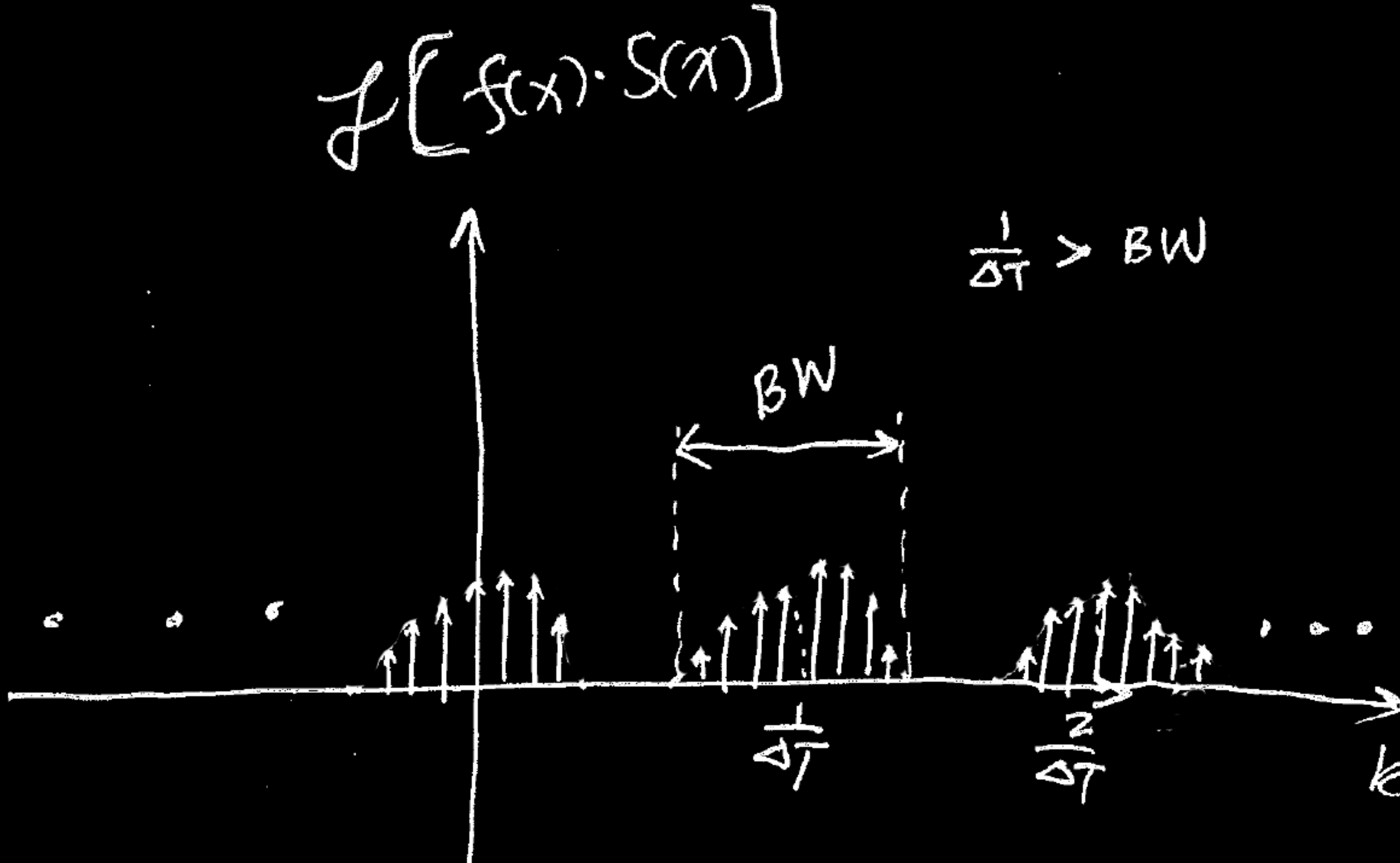


Digital FT



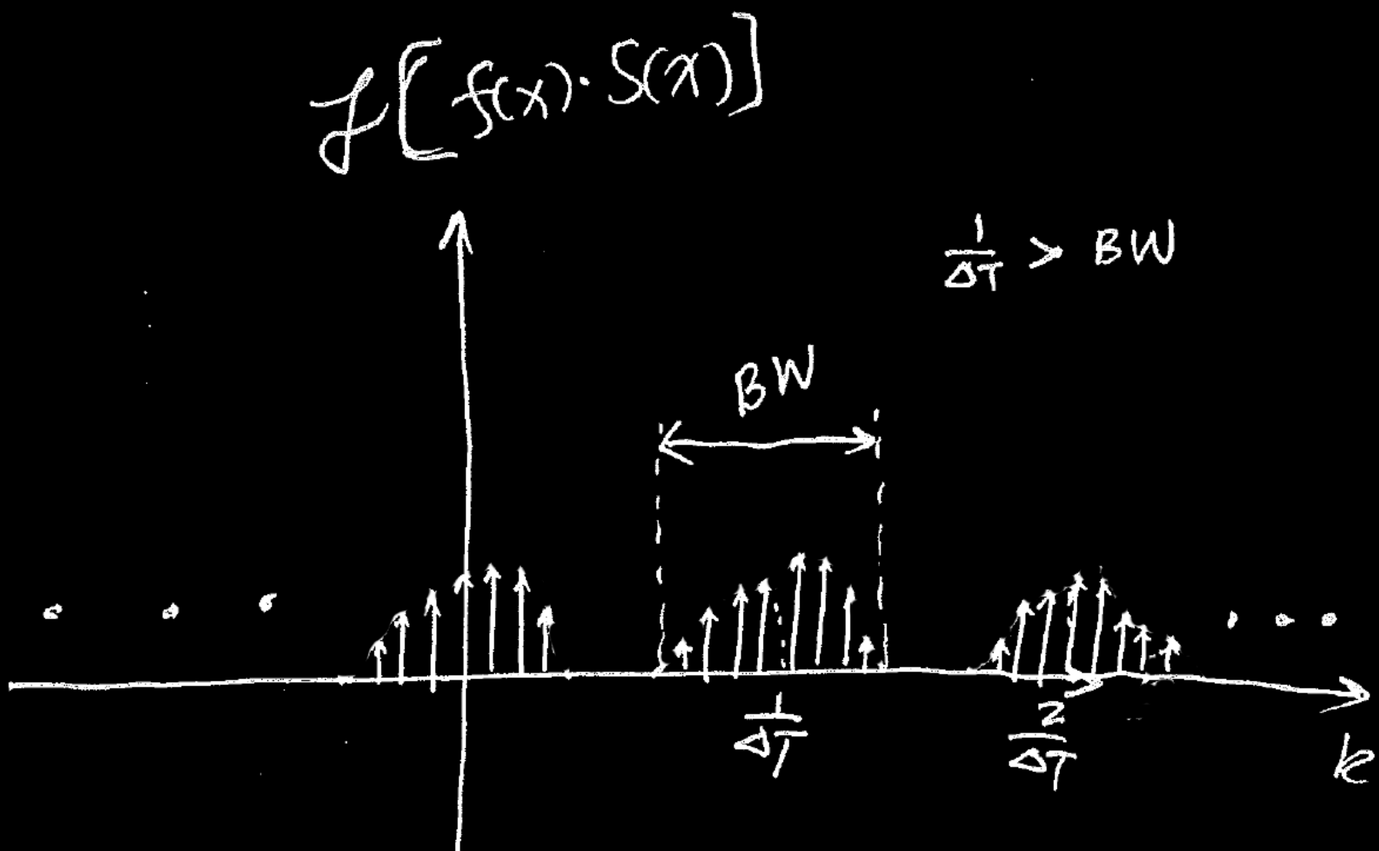
$\mathcal{F}[f(x) \cdot S(x)]$
$\frac{1}{\Delta T} > BW$
BW
$\frac{1}{\Delta T}$
$\frac{2}{\Delta T}$
k

Digital FT

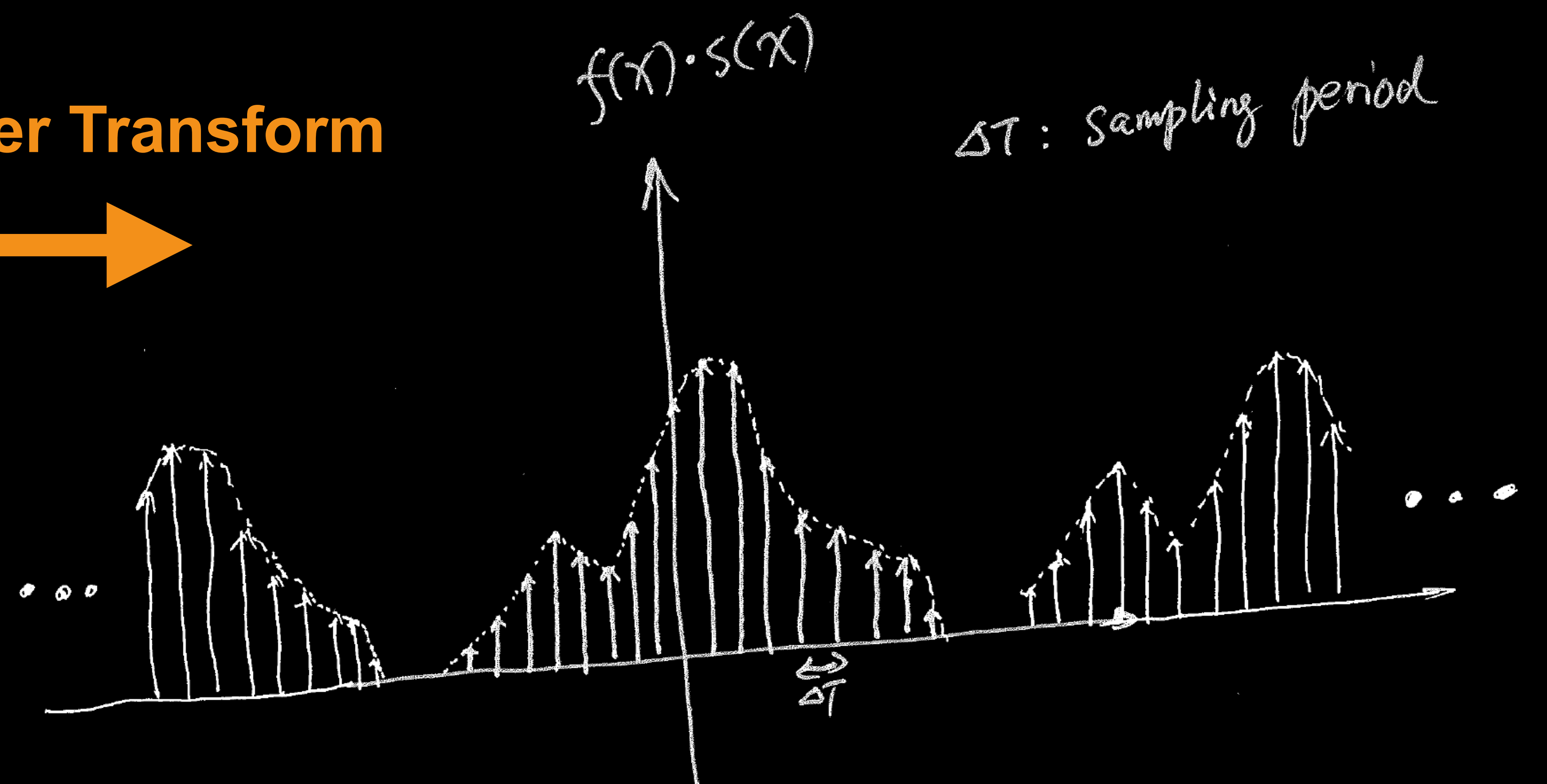


$\mathcal{F}[f(x) \cdot S(x)]$
$\frac{1}{\Delta T} > BW$
BW
$\frac{1}{\Delta T}$
$\frac{2}{\Delta T}$
k

Digital FT



Inverse Fourier Transform →



Summary of FT Properties

Fundamental Math Summary.pdf

FFT & FFTW
MATLAB DEMO

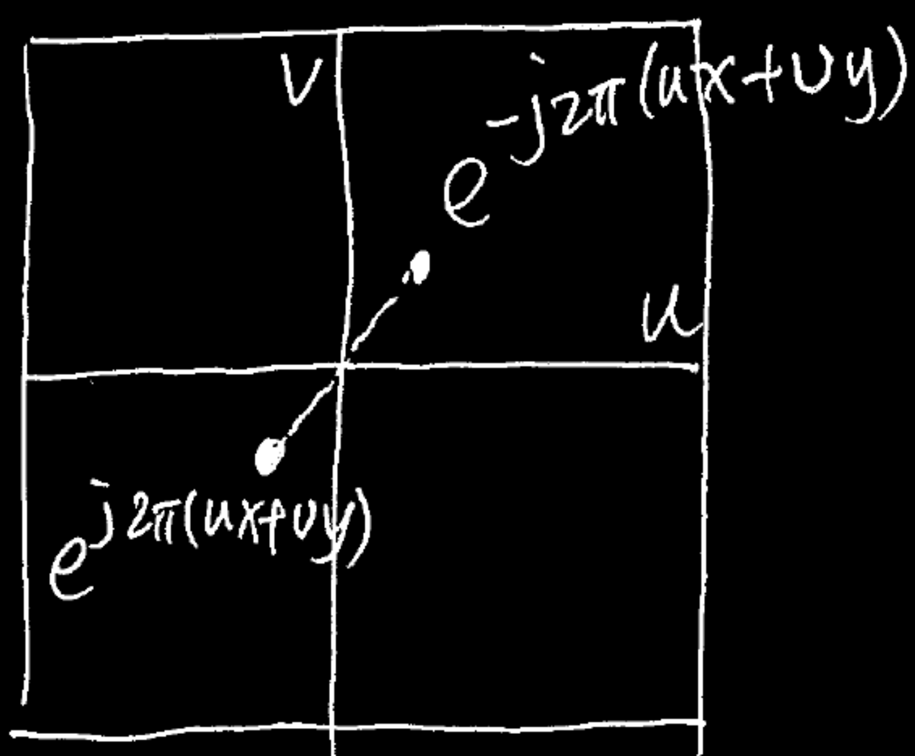
2D FT

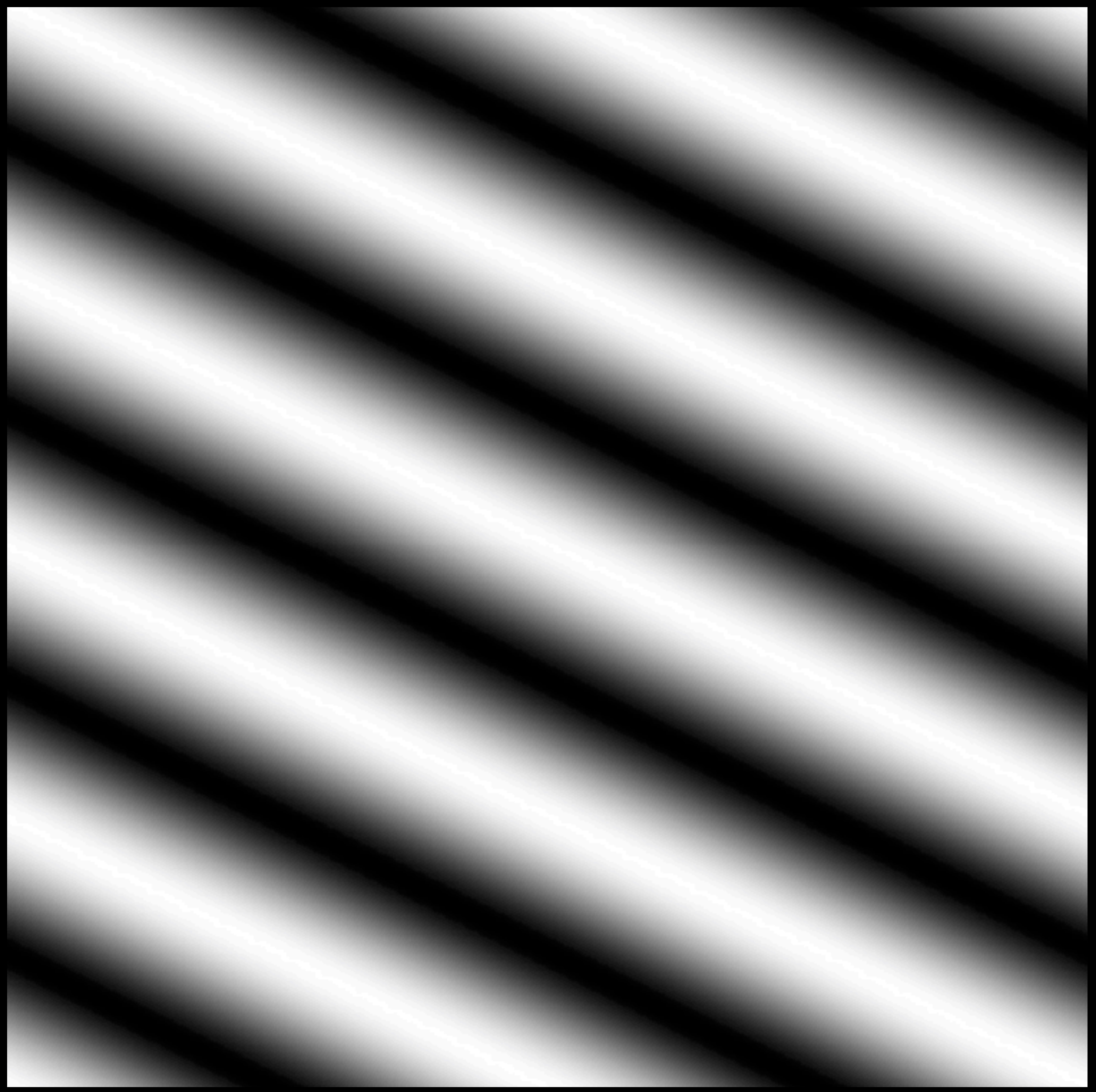
$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}\,du\,dv$$

2D FT

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy$$
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}\,du\,dv$$





2D FT

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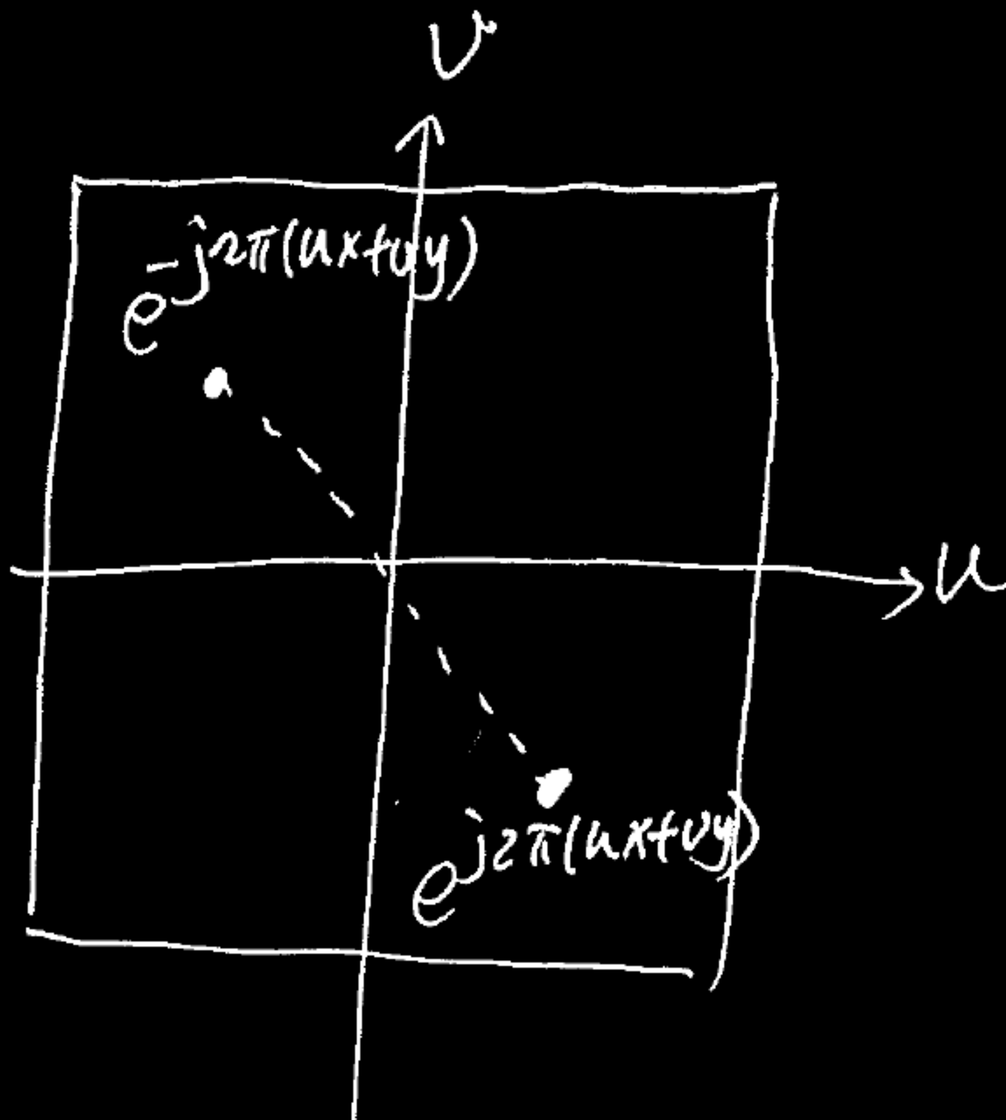
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}\,du\,dv$$



2D FT

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy$$

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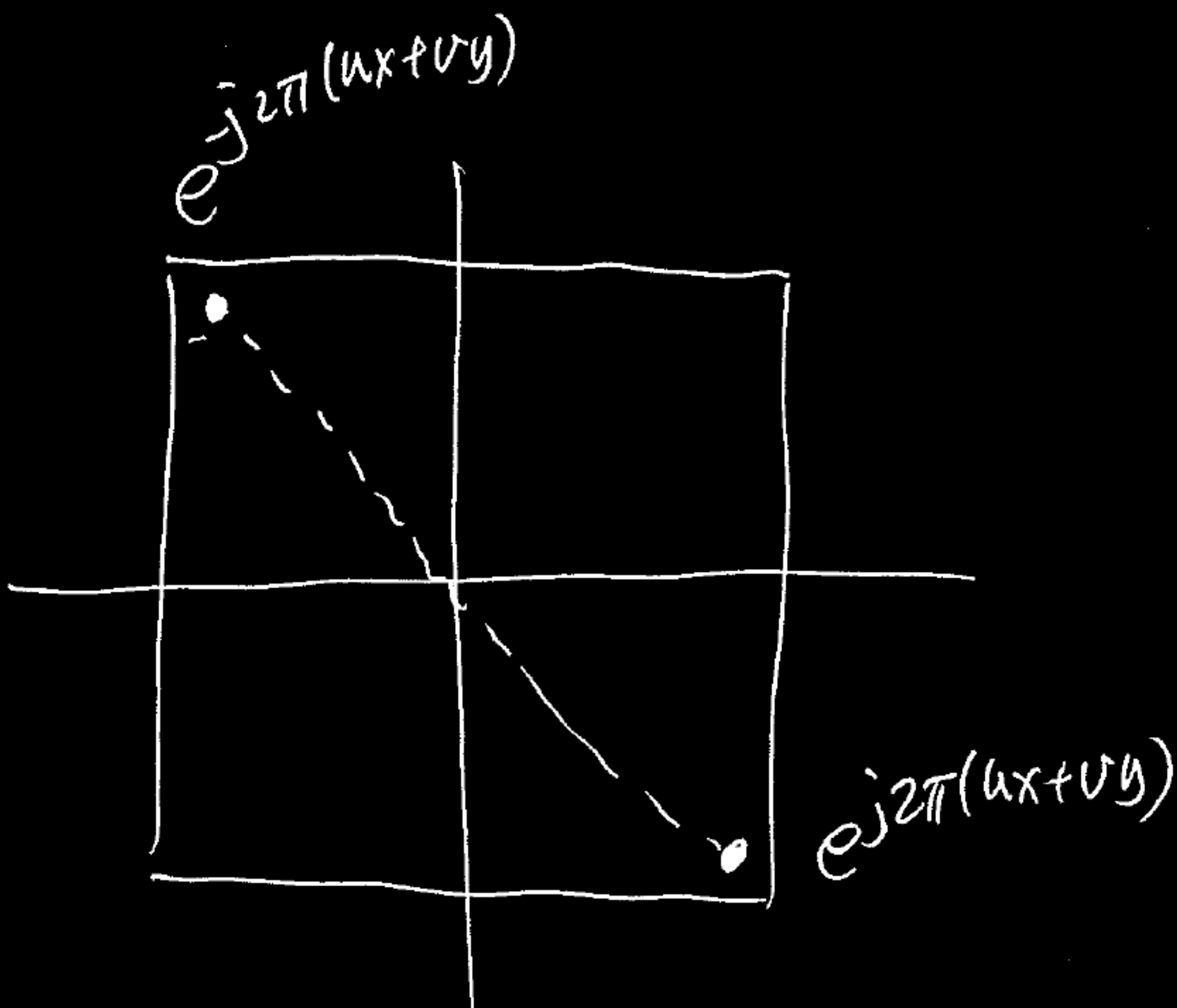
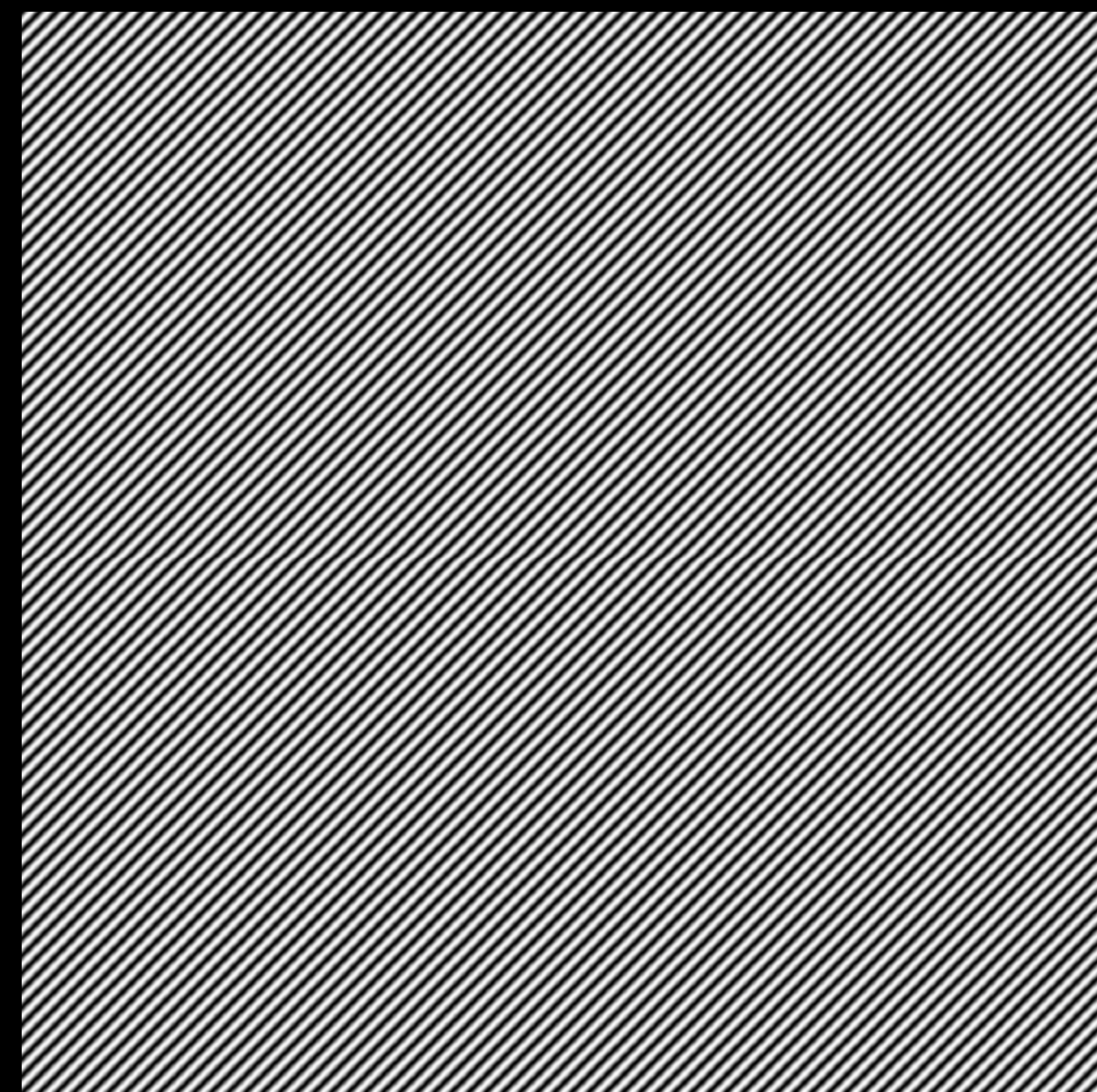
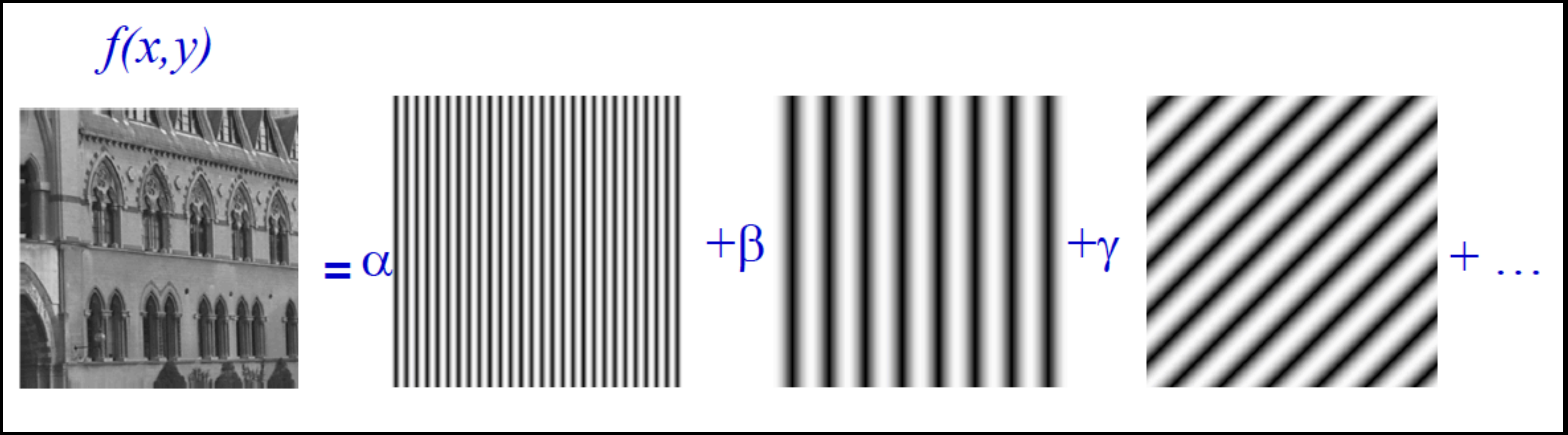
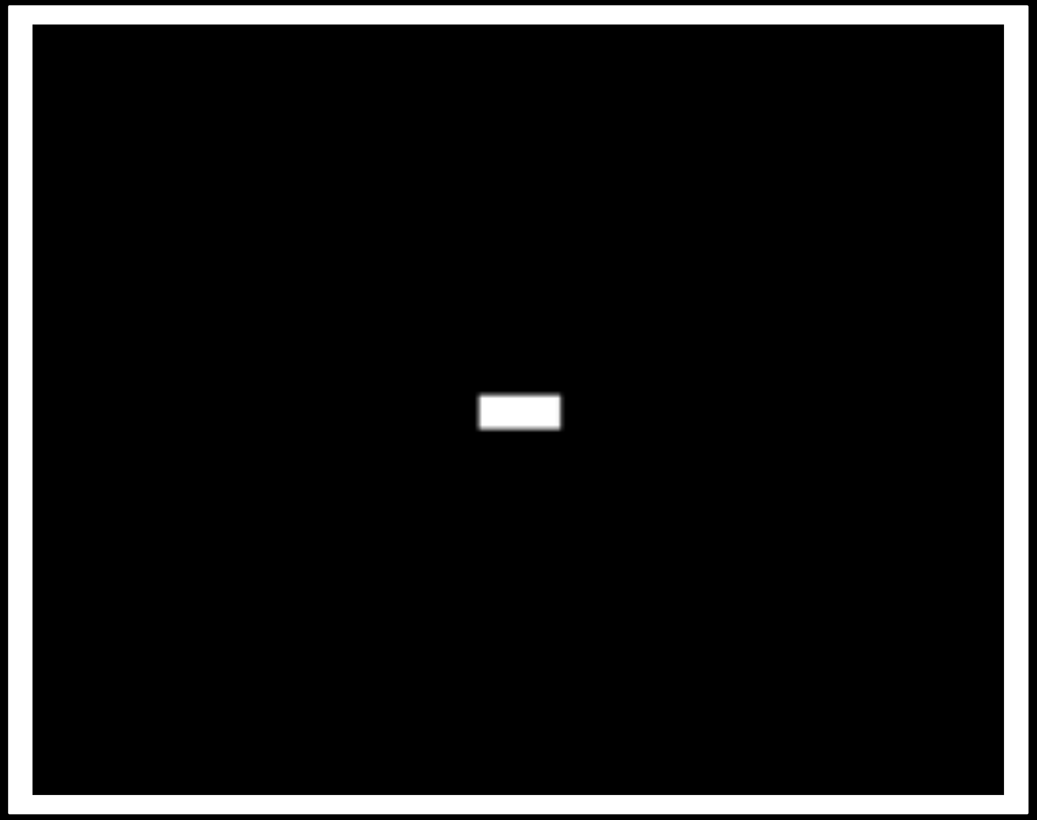


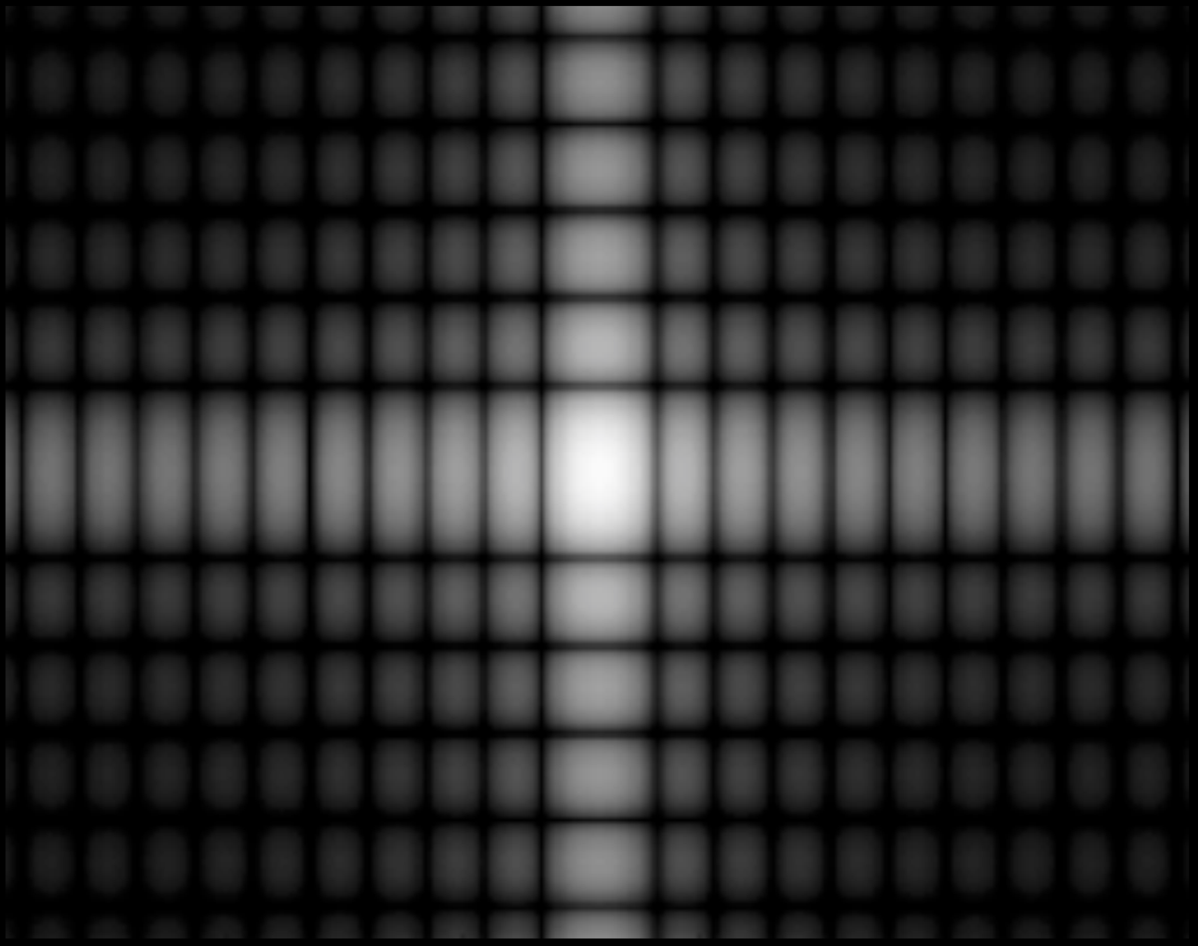


2D FT of 2D Image



2D FT of 2D Rect Function





2D FT of 2D Delta Function

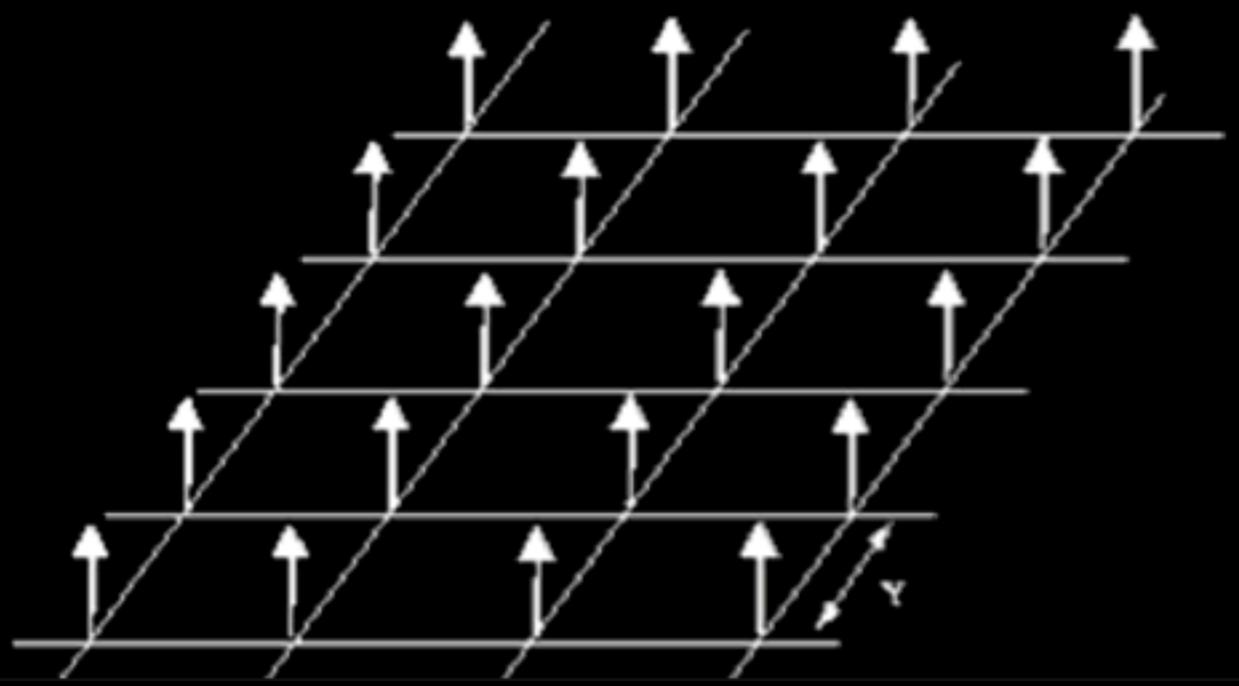
$$f(x,y) = \delta(x,y) = \delta(x)\delta(y)$$

$$F(u,v) = \int\int \delta(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$= 1$$

2D FT of 2D Comb Function

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x - nX)\delta(y - mY)$$



2D Convolution

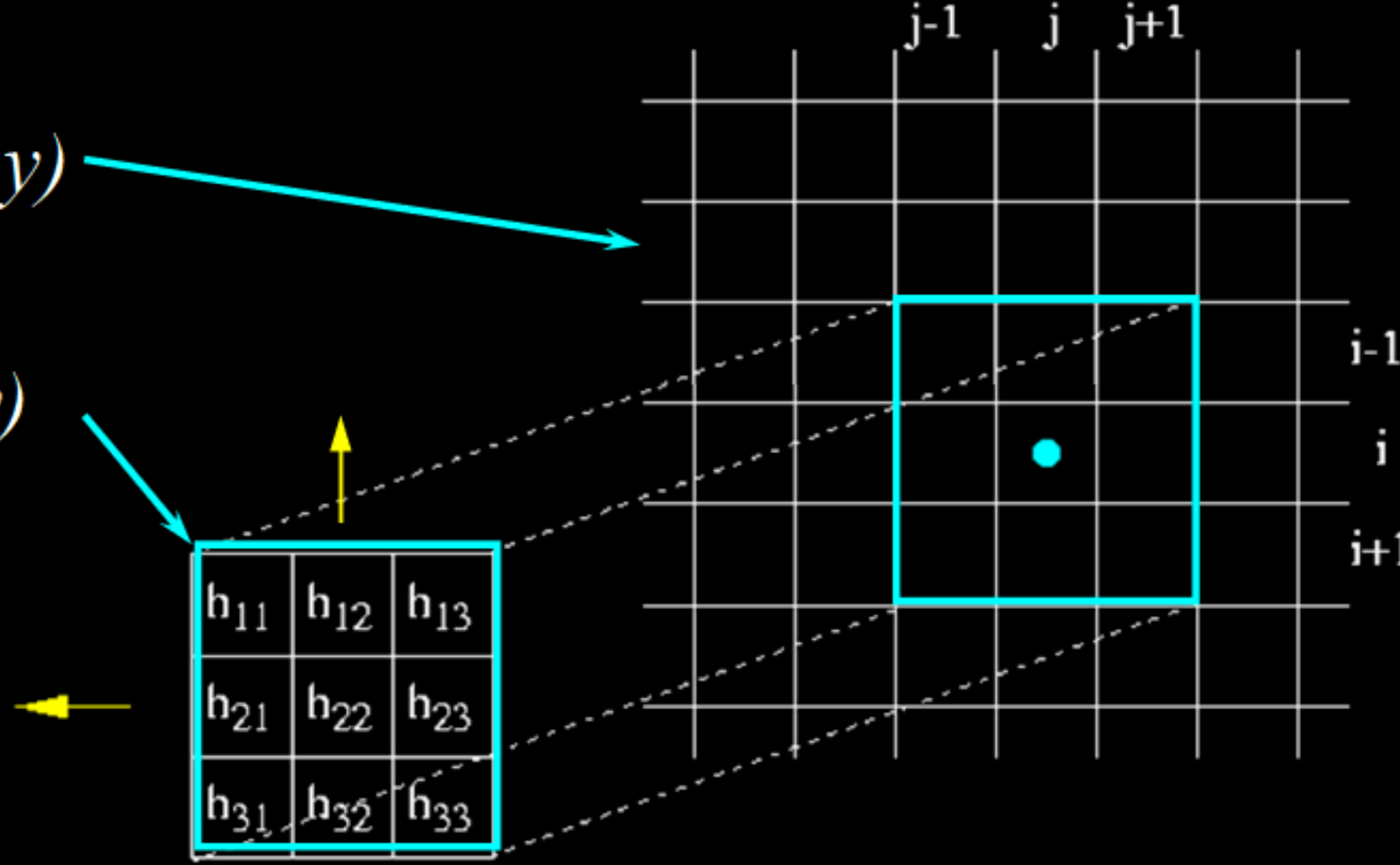
convolution

$$g(x,y) = h(x,y) * f(x,y) = f(x,y) * h(x,y)$$
$$= \int\int f(u,v)h(x-u,y-v)\,du\,dv$$

filtering

image $f(x,y)$

filter / kernel $h(x,y)$



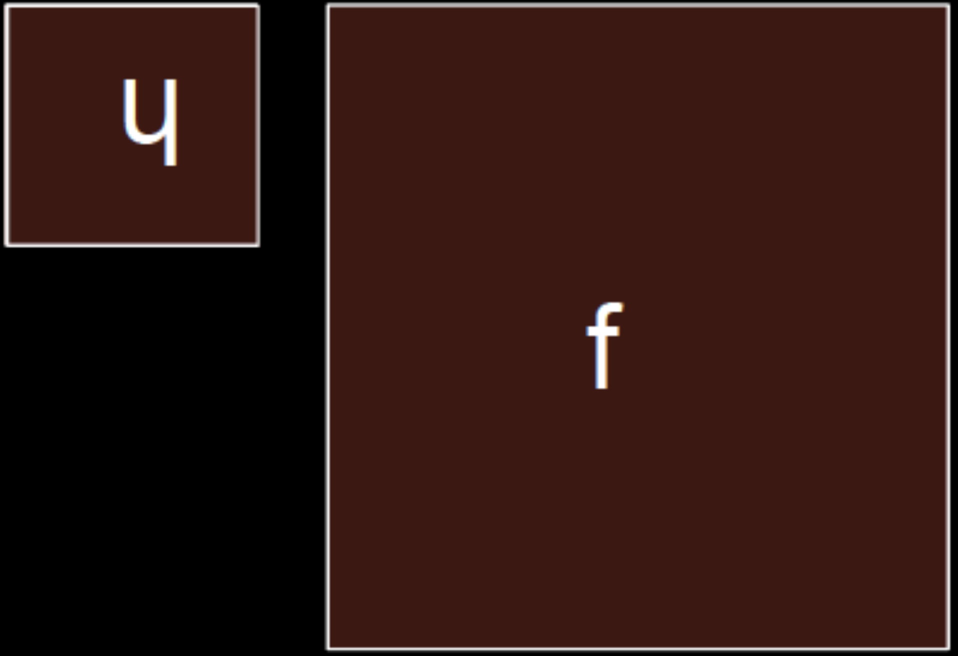
$$\begin{aligned} g(x,y) = \; & h_{11}\,f(i-1,j-1) + h_{12}\,f(i-1,j) + h_{13}\,f(i-1,j+1) + \\ & h_{21}\,f(i,j-1) + h_{22}\,f(i,j) + h_{23}\,f(i,j+1) + \\ & h_{31}\,f(i+1,j-1) + h_{32}\,f(i+1,j) + h_{33}\,f(i+1,j+1) \end{aligned}$$

for convolution, reflect filter in x and y axes

2D Convolution

$$g[i,j] \;=\; \sum_{u=-k}^{k} \sum_{v=-k}^{k} h[u,v]\,f[i-u,j-v]$$

convolution with h



2D Convolution Theorem

$$f(x, y) * h(x, y) \Leftrightarrow F(u, v)H(u, v)$$

Space convolution = frequency multiplication

2D FFT MATLAB DEMO

Questions?

- Related reading materials
  - Liang/Lauterbur - Chap 2.3, 2.4, 2.5
  - Nishimura - Chap 2.2, 2.4

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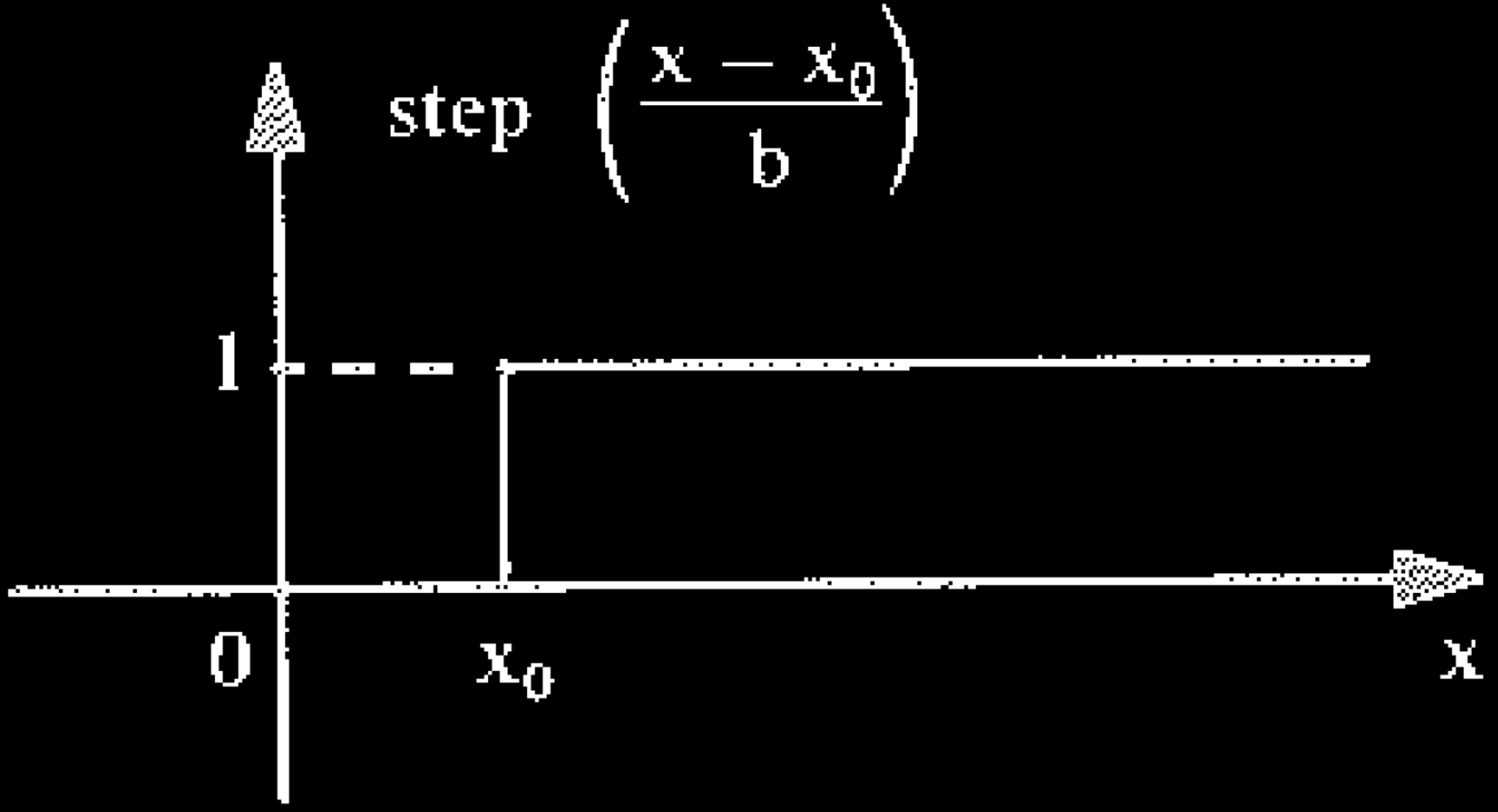
http://mrrl.ucla.edu/sunglab

Special Functions

$$\mathrm{step}\left(\frac{x-x_0}{b}\right)=\begin{cases}0, & \frac{x}{b}<\frac{x_0}{b}\\ \frac{1}{2}, & \frac{x}{b}=\frac{x_0}{b}\\ 1, & \frac{x}{b}>\frac{x_0}{b}\end{cases}.$$

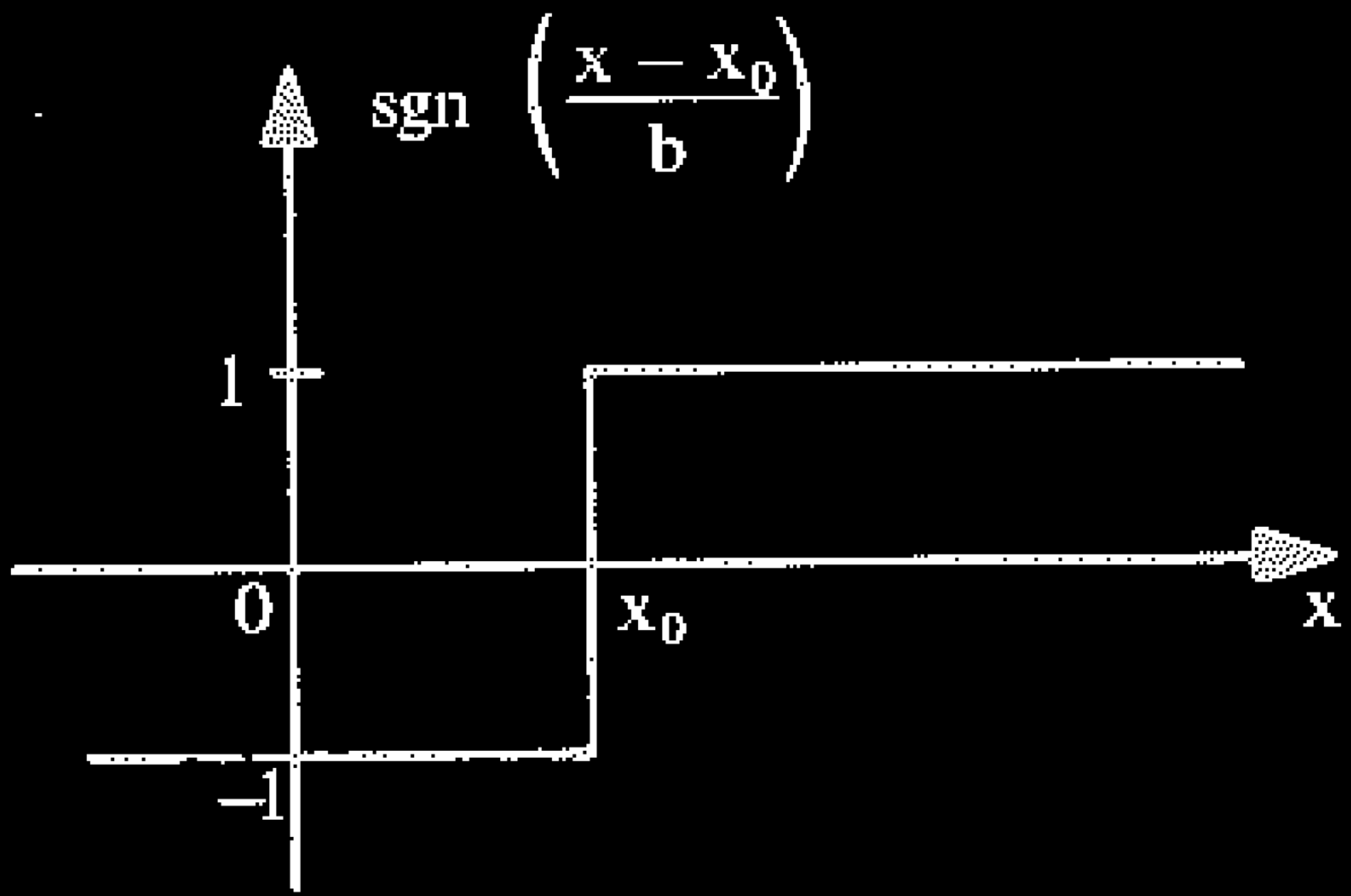
Step Function



Special Functions

$$\operatorname{sgn}\left(\frac{x-x_0}{b}\right)=\begin{cases}-1, & \frac{x}{b}<\frac{x_0}{b}\\ 0, & \frac{x}{b}=\frac{x_0}{b}\\ 1, & \frac{x}{b}>\frac{x_0}{b}\end{cases}.$$

Sign Function

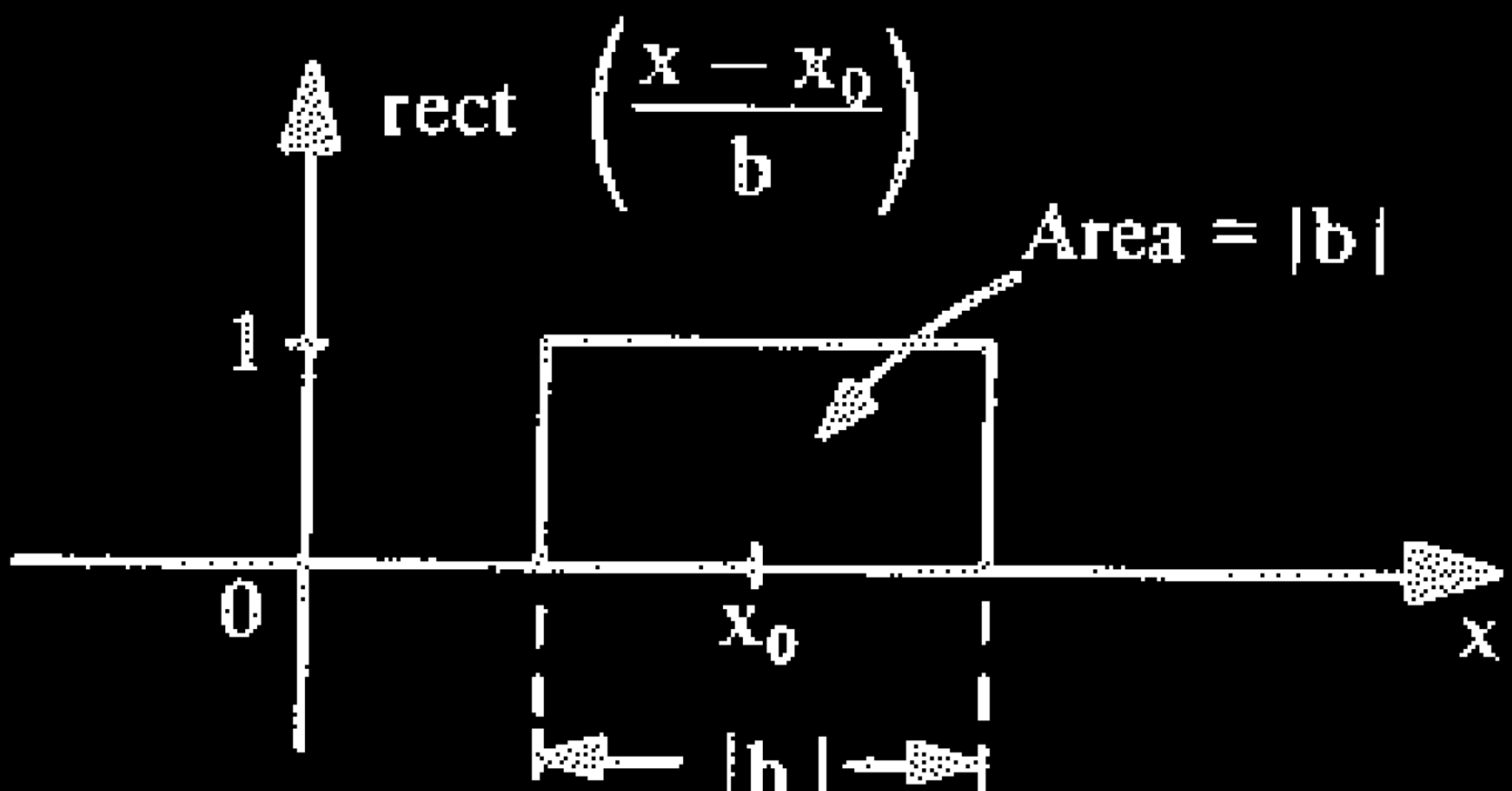


Special Functions

Rect Function

$$\mathrm{rect}\left(\frac{x-x_0}{b}\right)=\begin{cases}0, & \left|\frac{x-x_0}{b}\right|>\frac{1}{2}\\ \frac{1}{2}, & \left|\frac{x-x_0}{b}\right|=\frac{1}{2}\\ 1, & \left|\frac{x-x_0}{b}\right|<\frac{1}{2}\end{cases}.$$

$$\Pi(x) \triangleq \begin{cases}1 & |x|<1/2\\ 0 & \text{otherwise}\end{cases}$$

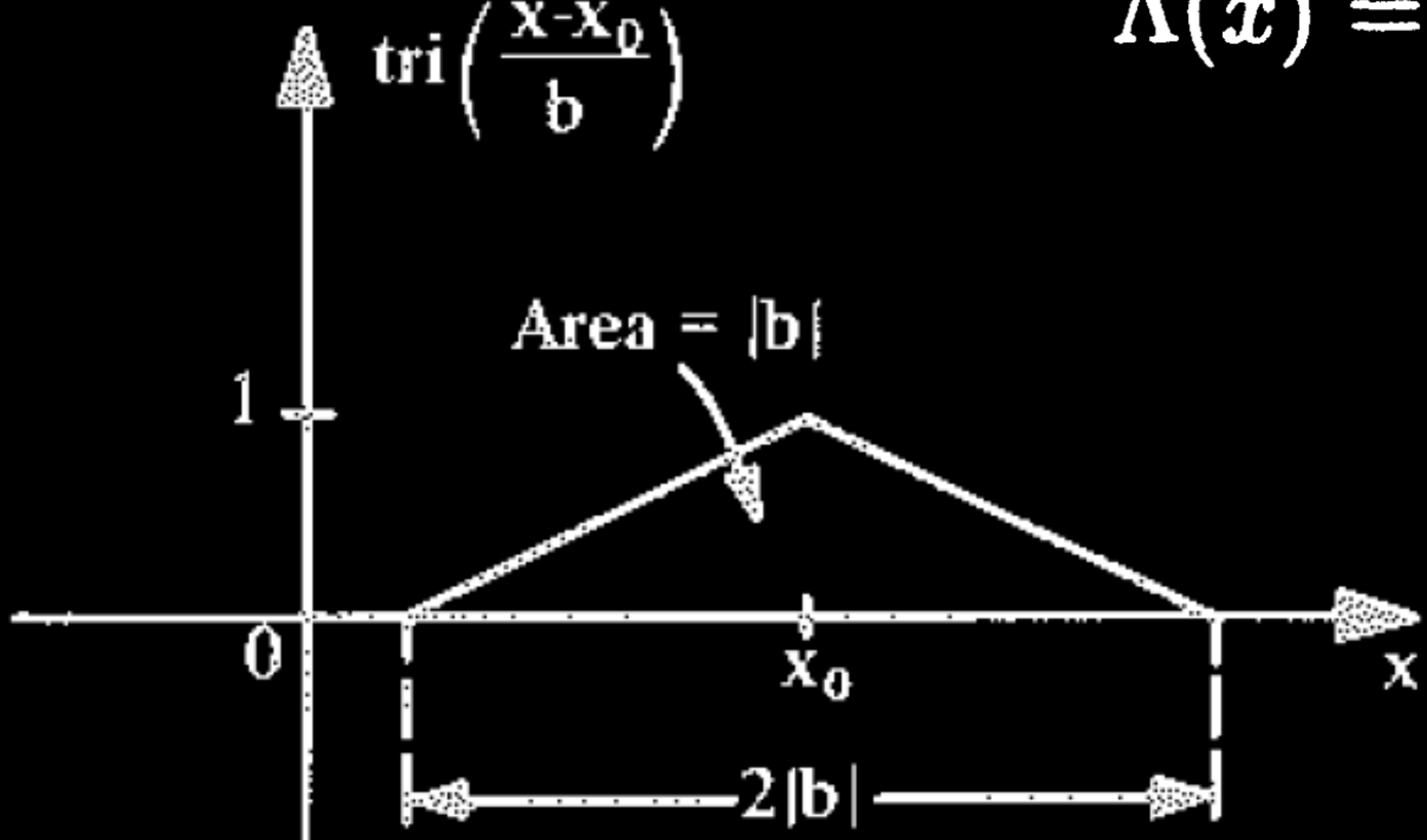


Special Functions

$$\operatorname{tri}\left(\frac{x-x_0}{b}\right)=\begin{cases} 0, & \left|\frac{x-x_0}{b}\right| \geqslant 1 \\ 1-\left|\frac{x-x_0}{b}\right|, & \left|\frac{x-x_0}{b}\right| < 1 \end{cases}.$$

Triangular Function

$$\Lambda(x) \triangleq \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



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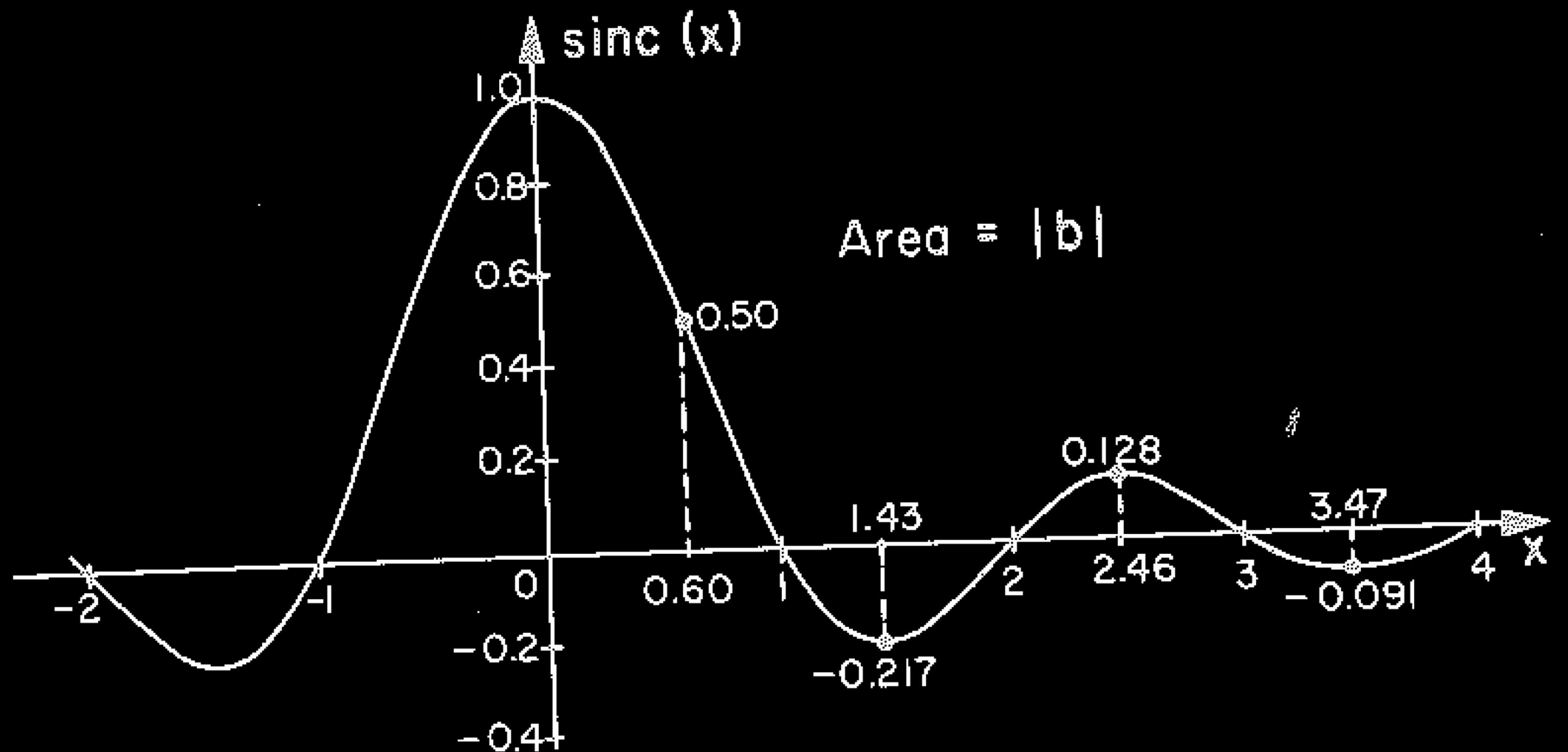
Special Functions

$$\operatorname{sinc}\left(\frac{x-x_0}{b}\right)=\frac{\sin\pi\left(\frac{x-x_0}{b}\right)}{\pi\left(\frac{x-x_0}{b}\right)}.$$

Sinc Function



Special Functions

$Sinc^2$ Function

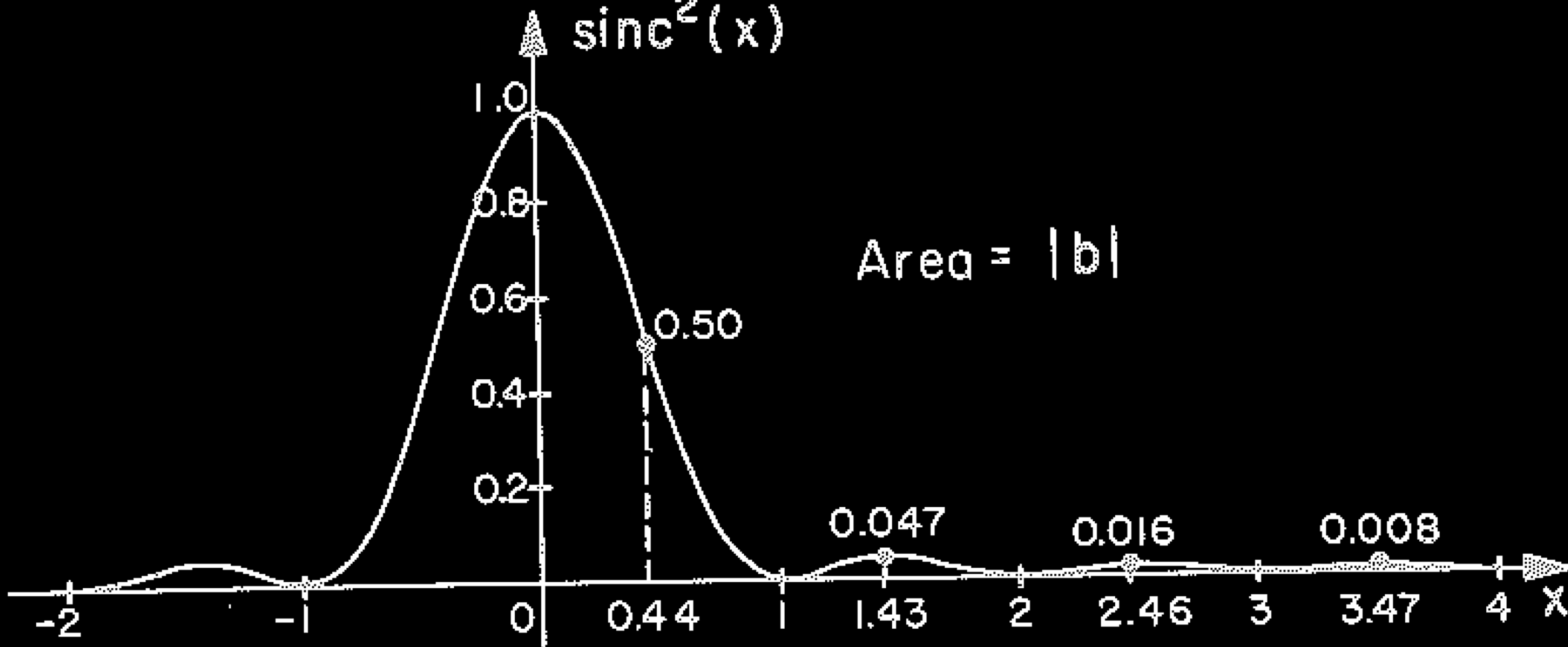
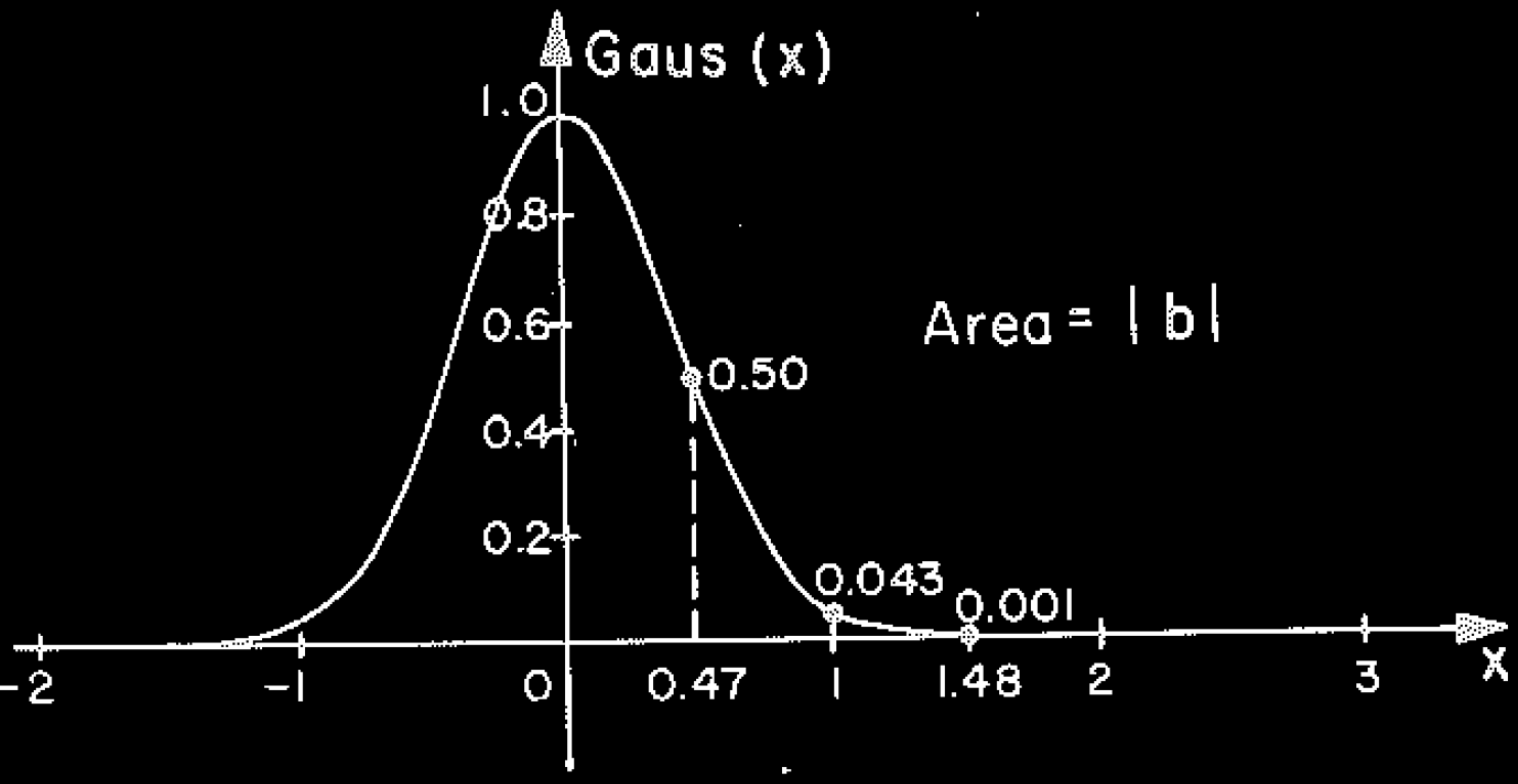


Figure 3-8 The $sinc^2$ function.

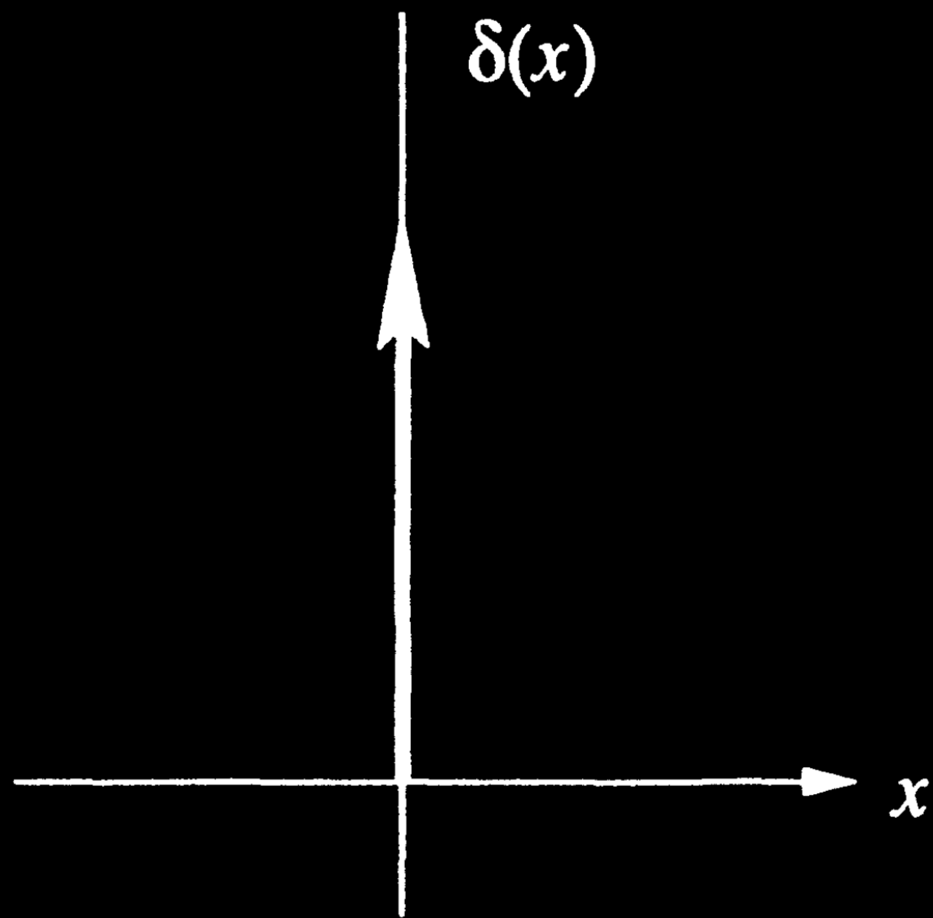
Special Functions

$$\mathrm{Gaus}\left(\frac{x-x_0}{b}\right)=\exp\left[-\pi\left(\frac{x-x_0}{b}\right)^2\right]$$

Gaussian Function



Delta (Impulse) Function



$$\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Pi\left(\frac{x}{\Delta x}\right)$$

$$\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \mathrm{sinc}\left(\frac{\pi x}{\Delta x}\right)$$

$$\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Lambda\left(\frac{x}{\Delta x}\right)$$

$\delta(x) = 0$ for $x \neq 0$

$\delta(x)$ is unbounded at $x = 0$

$\int_{-\infty}^{\infty} \delta(x)dx = 1$

Properties of Delta Function

Scaling property:

could be confusing!

$$\delta(ax) = \frac{1}{|a|}\delta(x) \qquad a \neq 0 \tag{2.45}$$

Sampling property:

$$\varphi(x)\delta(x - x_0) = \varphi(x_0)\delta(x - x_0) \tag{2.46a}$$

$$\int_{-\infty}^{\infty} \varphi(x)\delta(x - x_0)dx = \varphi(x_0) \tag{2.46b}$$

Derivative property:

$$\int_{-\infty}^{\infty} \varphi(x)\delta^{(n)}(x - x_0)dx = (-1)^n\varphi^{(n)}(x_0) \tag{2.47}$$

Why Delta Function is Important?

Impulse Response can fully characterize many linear systems

Any arbitrary function can be decomposed into a linear combination of delta functions

Relatives of Delta Function is essential in Fourier analysis of MRI signal/k-space

Relatives of Delta Function

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Kronecker Delta Function

Functions similarly in discrete systems as Delta Function

$$\sum_{n=-\infty}^{\infty} \varphi[n]\delta[n-n_0] = \varphi[n_0]$$

Relatives of Delta Function

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

Comb Function

$$\text{III}$$

$$\sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) = \frac{1}{\Delta x}\mathrm{comb}\left(\frac{x}{\Delta x}\right)$$