# Biomedical Imaging Systems Theory

Peng Hu, Winter 2019

The Fourier Transform:

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi sx}dx$$

The Inverse Fourier Transform:

$$f(x) = \int_{-\infty}^{\infty} F(s)e^{j2\pi sx}ds$$

**Symmetry Properties**:
If $g(x)$ is real valued, then $G(s)$ is Hermitian:

$$G(-s) = G^*(s)$$

If $g(x)$ is imaginary valued, then $G(s)$ is Anti-Hermitian:

$$G(-s) = -G^*(s)$$

In general:

$$g(x) = e(x) + o(x) = e_R(x) + ie_I(x) + o_R(x) + io_I(x)$$
$$G(s) = E(s) + O(s) = E_R(s) + iE_I(s) + iO_I(s) + O_R(s)$$

**Convolution**:

$$(g * h)(x) \triangleq \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi$$

**Autocorrelation**: Let $g(x)$ be a function satisfying $\int_{-\infty}^{\infty} |g(x)|^2 dx < \infty$ (finite energy) then

$$\gamma_g(x) \triangleq (g \star g^*)(x) \triangleq \int_{-\infty}^{\infty} g(\xi)g^*(\xi - x)d\xi = g(x) * g^*(-x)$$

**Cross correlation**: Let $g(x)$ and $h(x)$ be functions with finite energy. Then

$$(g \star h^*)(x) \triangleq \int_{-\infty}^{\infty} g(\xi + x)h^*(\xi)d\xi = \int_{-\infty}^{\infty} g(\xi)h^*(\xi - x)d\xi = (h \star g^*)^*(-x)$$

## The Delta Function: $\delta(x)$

- Scaling: $\delta(ax) = \frac{1}{|a|}\delta(x)$
- Sifting: $\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$

  $\int_{-\infty}^{\infty} \delta(x)f(x+a)dx = f(a)$
- Convolution: $\delta(x) * f(x) = f(x)$
- Product: $h(x)\delta(x) = h(0)\delta(x)$
- $\delta^2(x)$ - no meaning
- $\delta(x) * \delta(x) = \delta(x)$
- Fourier Transform of $\delta(x)$: $\mathcal{F}\{\delta(x)\} = 1$
- Derivatives:
  - $\int_{-\infty}^{\infty} \delta^{(n)}(x)f(x)dx = (-1)^n f^{(n)}(0)$
  - $\delta'(x) * f(x) = f'(x)$
  - $x\delta(x) = 0$
  - $x\delta'(x) = -\delta(x)$
- Meaning of $\delta[h(x)]$:

$$\delta[h(x)] = \sum_i \frac{\delta(x - x_i)}{|h'(x_i)|}$$

## The comb function

- Sampling: $\text{comb}(x)g(x) = \sum_{n=-\infty}^{\infty} g(n)\delta(x-n)$
- Replication: $\text{comb}(x) * g(x) = \sum_{n=-\infty}^{\infty} g(x-n)$
- Fourier Transform: $\mathcal{F}\{\text{comb}(x)\} = \text{comb}(s)$
- Scaling: $\text{comb}(ax) = \sum \delta(ax - n) = \frac{1}{|a|}\sum \delta(x - \frac{n}{a})$

## Even and Odd Impulse Pairs

Even: $\delta\delta(x) = \delta(x+1) + \delta(x-1)$

Odd: $\delta_\delta(x) = \delta(x+1) - \delta(x-1)$

Fourier Transforms: $\mathcal{F}\{\frac{1}{2}\delta\delta(x)\} = \cos 2\pi s$

$\mathcal{F}\{\frac{1}{2}\delta_\delta(x)\} = j\sin 2\pi s$

## Fourier Transform Theorems

- Linearity: $\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha F(s) + \beta G(s)$
- Similarity: $\mathcal{F}\{g(ax)\} = \frac{1}{|a|}G(\frac{s}{a})$
- Shift: $\mathcal{F}\{g(x-a)\} = e^{-j2\pi as}G(s)$

  $\mathcal{F}\{g(ax-b)\} = \frac{1}{|a|}e^{-j2\pi s\frac{b}{a}}G(\frac{s}{a})$
- Rayleigh's: $\int_{-\infty}^{\infty} |g(x)|^2dx = \int_{-\infty}^{\infty} |G(s)|^2ds$
- Power: $\int_{-\infty}^{\infty} f(x)g^*(x)dx = \int_{-\infty}^{\infty} F(s)G^*(s)ds$
- Modulation:

$$\mathcal{F}\{g(x)cos(2\pi s_0x)\} = \frac{1}{2}[G(s-s_0) + G(s+s_0)]$$

- Convolution: $\mathcal{F}\{f * g\} = F(s)G(s)$
- Autocorrelation: $\mathcal{F}\{g \star g^*\} = |G(s)|^2$
- Cross Correlation: $\mathcal{F}\{f \star g^*\} = F(s)G^*(s)$
- Derivative:
  - $\mathcal{F}\{g'(x)\} = j2\pi s G(s)$
  - $\mathcal{F}\{g^{(n)}(x)\} = (j2\pi s)^n G(s)$
  - $\mathcal{F}\{x^n g(x)\} = (\frac{j}{2\pi})^n G^{(n)}(s)$
- Fourier Integral: If $g(x)$ is of bounded variation and is absolutely integrable, then

$$\mathcal{F}^{-1}\{\mathcal{F}\{g(x)\}\} = \frac{1}{2}[g(x^+) + g(x^-)]$$

- Moments:

$$\int_{-\infty}^{\infty} f(x)dx = F(0)$$

$$\int_{-\infty}^{\infty} xf(x)dx = \frac{j}{2\pi}F'(0)$$

$$\int_{-\infty}^{\infty} x^n f(x)dx = (\frac{j}{2\pi})^n F^{(n)}(0)$$

- Miscellaneous:
  If $\mathcal{F}\{g(x)\} = G(s)$ then $\mathcal{F}\{G(x)\} = g(-s)$
  and $\mathcal{F}\{g^*(x)\} = G^*(-s)$

$$\mathcal{F}\left\{\int_{-\infty}^{x} g(\xi)d\xi\right\} = \tfrac{1}{2}G(0)\delta(s) + \frac{G(s)}{j2\pi s}$$

## Central Limit Theorem

Given a function $f(x)$, if $F(s)$ has a single absolute maximum at $s = 0$; and, for sufficiently small $|s|$, $F(s) \approx a - cs^2$ where $0 < a < \infty$ and $0 < c < \infty$, then:

$$\lim_{n\to\infty} \frac{[\sqrt{n}f(\sqrt{n}x)]^{*n}}{a^n} = \sqrt{\frac{\pi a}{2}}e^{-\frac{\pi a}{c}x^2}$$

and

$$[f(x)]^{*n} \approx \frac{a^{n+\frac{1}{2}}}{n^{\frac{1}{2}}}\sqrt{\frac{\pi}{c}}e^{-\frac{\pi a}{cn}x^2}$$

## Linear Systems

For a linear system $w(t) = \mathcal{S}[v(t)]$ with response $h(t,\tau)$ to a unit impulse at shift $\tau$:

$$\mathcal{S}[\alpha v_1(t) + \beta v_2(t)] = \alpha\mathcal{S}[v_1(t)] + \beta\mathcal{S}[v_2(t)]$$

$$w(t) = \int_{-\infty}^{\infty} v(\tau)h(t,\tau)d\tau$$

If such a system is shift-invariant, then:

$$w(t-\tau) = \mathcal{S}[v(t-\tau)]$$

and

$$\begin{aligned} w(t) &= \int_{-\infty}^{\infty} v(\tau)h(t-\tau)d\tau \\ &= (v * h)(t) \end{aligned}$$

The eigenfunctions of any linear shift-invariant system are $e^{j2\pi f_0 t}$, since for a system with transfer function H(s), the response to an input of $v(t) = e^{j2\pi f_0 t}$ is given by: $w(t) = H(f_0)e^{j2\pi f_0 t}$.

## Sampling Theory

$$\begin{aligned} \hat{g}(x) &= \frac{1}{X}\text{comb}(\frac{x}{X})g(x) \\ &= \sum_{n=-\infty}^{\infty} g(nX)\delta(x - nX) \end{aligned}$$

$$\begin{aligned} \hat{G}(s) &= \text{comb}(Xs) * G(s) \\ &= \frac{1}{X}\sum_{n=-\infty}^{\infty} G(s - \frac{n}{X}) \end{aligned}$$

Whittaker-Shannon-Kotelnikov Theorem: For a bandlimited function $g(x)$ with cutoff frequencies $\pm s_c$, and with no discrete sinusoidal components at frequency $s_c$,

$$g(x) = \sum_{n=-\infty}^{\infty} g(\frac{n}{2s_c})\text{sinc}[2s_c(x - \frac{n}{2s_c})]$$

## Fourier Tranforms for Periodic Functions

For a function $p(x)$ with period $L$, let $f(x) = p(x)\text{rect}(\frac{x}{L})$. Then

$$p(x) = f(x) * \sum_{n=-\infty}^{\infty} \delta(x - nL)$$

$$P(s) = \frac{1}{L}\sum_{n=-\infty}^{\infty} F(\frac{n}{L})\delta(s - \frac{n}{L})$$

The complex Fourier series representation:

$$p(x) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi\frac{n}{L}x}$$

where

$$\begin{aligned} c_n &= \frac{1}{L}F(\frac{n}{L}) \\ &= \frac{1}{L}\int_{-L/2}^{L/2} p(x)e^{-j2\pi\frac{n}{L}x}dx \end{aligned}$$

## The Discrete Fourier Transform

Let $g(x)$ be a physical process, and let $f(x) = g(x)$ for $0 \leq x \leq 2L$, $f(x) = 0$ otherwise. Suppose f(x) is approximately bandlimited to $\pm B$ Hz, so we sample $f(x)$ every $1/2B$ seconds, obtaining $N = \lfloor 4BL \rfloor$ samples.
The Discrete Fourier Transform:

$$F_m = \sum_{n=0}^{N-1} f_n e^{-j\frac{2\pi mn}{N}} \quad \text{for } m = 0, \ldots, N-1$$

The Inverse Discrete Fourier Transform:

$$f_n = \frac{1}{N} \sum_{m=0}^{N-1} F_m e^{j\frac{2\pi mn}{N}} \quad \text{for } n = 0, \ldots, N-1$$

**Convolution**:

$$h_n = \sum_{k=0}^{N-1} f_k g_{n-k} \quad \text{for } n = 0, \ldots, N-1 \quad \text{where } f,\ g \text{ are periodic}$$

**Serial Product**:

$$h_n = \sum_{k=0}^{N-1} f_k g_{n-k} \quad \text{for } n = 0, \ldots, 2N-2 \quad \text{where } f,\ g \text{ are not periodic}$$

### DFT Theorems

- Linearity: $\mathcal{DFT}\{\alpha f_n + \beta g_n\} = \alpha F_m + \beta G_m$
- Shift: $\mathcal{DFT}\{f_{n-k}\} = F_m e^{-j\frac{2\pi}{N}km}$ ($f$ periodic)
- Parseval's: $\sum_{n=0}^{N-1} f_n g_n^* = \frac{1}{N}\sum_{m=0}^{N-1} F_m G_m^*$
- Convolution: $F_m G_m = \mathcal{DFT}\{\sum_{k=0}^{N-1} f_k g_{n-k}\}$

### The Two-Dimensional Fourier Transform

$$F(s_x, s_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(s_x x + s_y y)} dx dy$$

The Inverse Two-Dimensional Fourier Transform:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(s_x, s_y) e^{j2\pi(s_x x + s_y y)} ds_x ds_y$$

The Hankel Transform (zero order):

$$F(\rho) = 2\pi \int_0^{\infty} f(r) J_0(2\pi r\rho) r dr$$

The Inverse Hankel Transform (zero order):

$$f(r) = 2\pi \int_0^{\infty} F(\rho) J_0(2\pi r\rho) \rho d\rho$$

**Central-Section Theorem**: The 1-D Fourier transform $G_\theta(s)$ of any projection $g_\theta(x')$ through $f(x,y)$ is identical with the 2-D transform $F(s_x, s_y)$ of $f(x,y)$, evaluated along a slice through the origin in the 2-D frequency domain, the slice being at angle $\theta$ to the x-axis. i.e.:

$$G_\theta(s) = F(s\cos\theta, s\sin\theta)$$

Reconstruction by $\rho$ filtering and Backprojection:

$$\begin{aligned} f(x,y) &= \int_0^{\pi} \mathcal{F}^{-1}\{\mid \rho \mid G_\theta(\rho)\} d\theta \\ &= \int_0^{\pi} \tilde{g}_\theta(x\cos\theta + y\sin\theta) d\theta \end{aligned}$$

*Original version compiled by John Jackson*