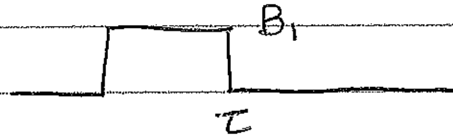
Ex 1 >




$B_1(t) = B_1 \quad , \quad 0 \le t \le \tau$

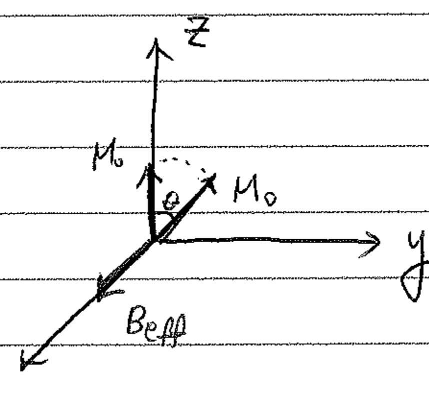
$$\vec{B}_{eff} = \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix} \quad \text{at on-resonance}$$

$$\frac{d\vec{M}_{ROT}}{dt} = \vec{M}_{ROT} \times \gamma \vec{B}_{eff}$$

$$\Longrightarrow \quad \vec{M}_{ROT}(t) = R_x(\gamma B_1 t) \begin{bmatrix} 0 \\ 0 \\ M_0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ M_0 \sin(\gamma B_1 \cdot t) \\ M_0 \cos(\gamma B_1 \cdot t) \end{bmatrix}$$

Graphically,




tip angle $\theta = \gamma \cdot B_1 \tau$

"tip angle" or "flip angle"

$$\theta = \int_0^{\tau} \gamma \cdot B(t')\, dt' = \gamma \cdot B_1 \tau$$

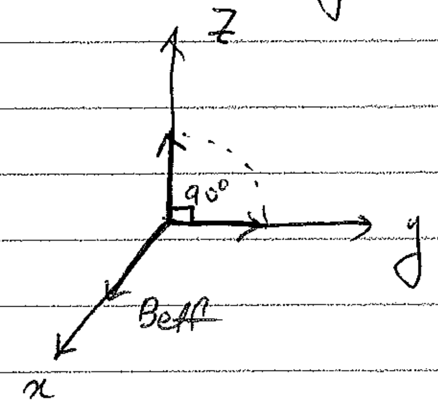
* Numbers

$$\theta = 90^\circ = \frac{\pi}{2} \quad , \quad \tau = 1\,ms$$

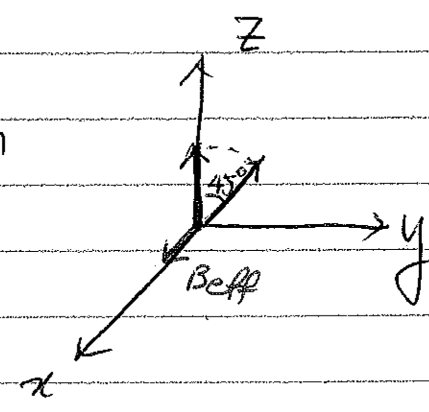
$$\frac{\pi}{2} = \gamma \cdot 1 \cdot B_1 \Rightarrow B_1 \simeq 0.06G = 6\mu T$$

Q : What if $B_1 = 12\,\mu T$

* $B_1$ inhomogeneity




50% reduction



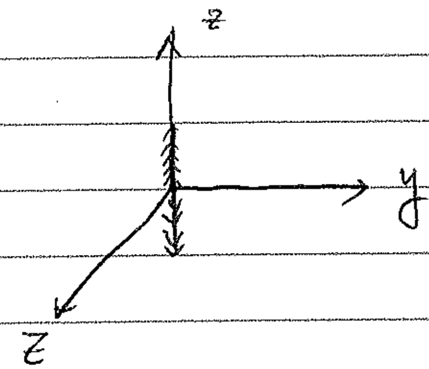
Ex 2) No RF pulse, only gradient along z

$$\vec{B} = (B_0 + G_z \cdot z)\,\hat{k}$$

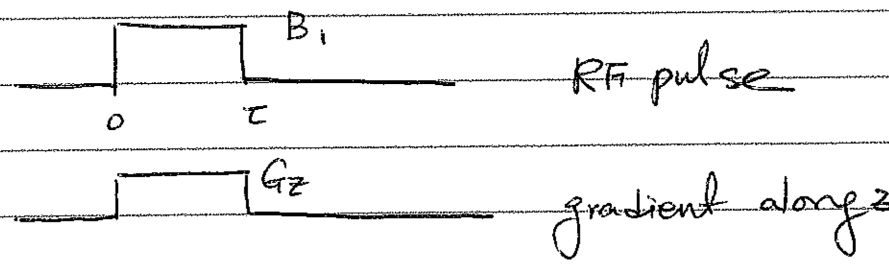
At on-resonance,

$$\vec{B}_{eff} = \left(B_0 + G_z \cdot z - \frac{\omega_{RF}}{\gamma}\right)\hat{k}$$

$$= G_z \cdot z\,\hat{k}$$




Ex 3)

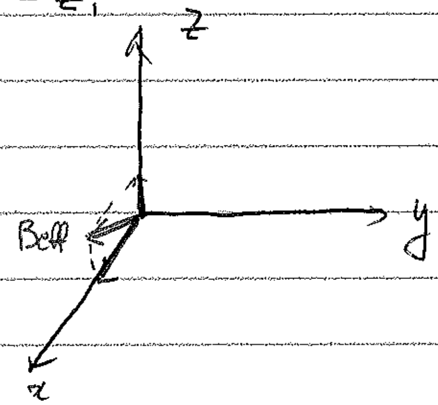



$$\vec{B}_{eff} = \left(B_0 + G_z \cdot z - \frac{\omega_{RF}}{\gamma}\right)\hat{k} + B_1\,\hat{i}$$

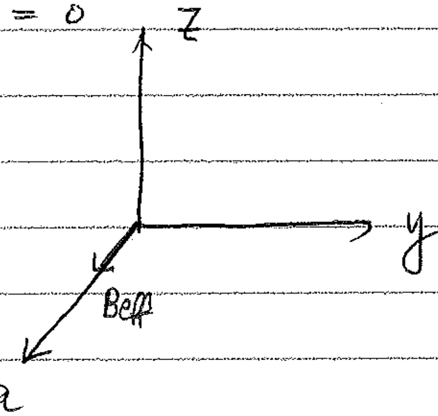
At on-resonance,

$$\vec{B}_{eff} = G_z \cdot z\,\hat{k} + B_1\,\hat{i}$$

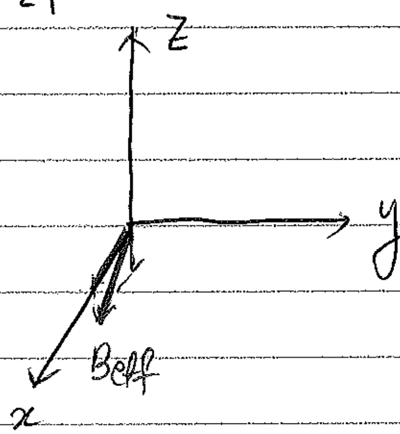
at $z = z_1$




at $z = 0$




at $z = -z_1$




$\Rightarrow$ if $z=0$ (or if $G_z = 0$), similar to non-selective case

$\Rightarrow$ other positions in $z$, different $B_{eff}(z)$ !

* Full analytical solution

$$\frac{d\vec{M}}{dt} = \begin{bmatrix} 0 & \omega(z) & 0 \\ -\omega(z) & 0 & \omega_1(t) \\ 0 & -\omega_1(t) & 0 \end{bmatrix} \vec{M}$$

$M_r = M_x + i M_y \leftarrow$ phasor representation

$$\frac{dM_r}{dt} = -i\,\omega(z) M_r + i\,\omega_1(t) M_z$$

$$\frac{dM_z}{dt} = -i\,\omega_1(t) M_y$$

Solution $\Longrightarrow$ very difficult $\Longrightarrow$ using approximation

$\sin\theta \simeq \theta$

$\cos\theta \simeq 1$

$M_z \simeq M_0 \simeq$ constant

$$\Rightarrow \quad \frac{dM_z}{dt} = 0$$

$$\frac{dM_r}{dt} = -i\,\omega(z) M_r + i\,\omega_1(t) M_0$$

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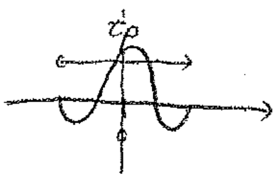
$$\frac{dy}{dx} + p(x)\,y = q(x)$$

$$y = \frac{\int u(x)\, q(x)\, dx}{u(x)}, \quad \text{where} \quad u(x) = \exp\left\{\int p(x)\,dx\right\}$$

$$\frac{d M_r^y}{dt_x} + \underbrace{i\, W(z)}_{p(x)} M_r^y = \underbrace{i\, W_1(t)\, M_0}_{\lambda(x)}$$

$$M_r(t,z) = i \cdot M_0\, e^{-iW(z)\cdot t} \int_0^t W_1(\tau)\, e^{-iW(z)\tau}\, d\tau$$

Assume, the RF pulse is symmetric and peaks at $t=0$ such that the pulse ends at $t = \tau_p/2$, $\tau_p$ is the length of the RF pulse $\rightarrow$ let $\tau' = \tau - \tau_p/2$




$$M_r(\tau_p) = i\, M_0\, e^{-\frac{i W(z) \tau_p}{2}} \int_{-\tau_p/2}^{\tau_p/2} W_1\left(t + \frac{\tau_p}{2}\right) e^{-i W(z) t}\, dt$$

assuming $W_1\left(t + \frac{\tau_p}{2}\right) = \begin{cases} 0 & \text{for } |t| > \frac{\tau_p}{2} \\ W_1\left(t + \frac{\tau_p}{2}\right) & \text{for } |t| < \frac{\tau_p}{2} \end{cases}$

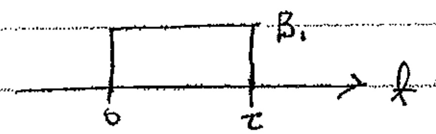
$$\Rightarrow M_r(\tau, z) = i\, M_0\, e^{-i W(z)\tau/2}\, \mathcal{F}_{1D}\left\{ W_1\left(t + \frac{\tau}{2}\right)\right\}\Bigg|_{f = -f(z) = -(\gamma/2\pi) G_z \cdot z}$$

⑦

$$* \quad |M_{xy}(\tau, z)| = M_0 \, \mathcal{F}_{1D}\left\{ W_1\left(t + \frac{\tau}{2}\right)\right\}\Big|_{f = -\frac{\gamma}{2\pi} G_z z}$$

* Small tip-angle example
  - consider a rectangular RF pulse (duration of $\tau$)

$$B_1(t) = B_1 \cdot \Pi\left(\frac{t-\tau/2}{\tau}\right)$$

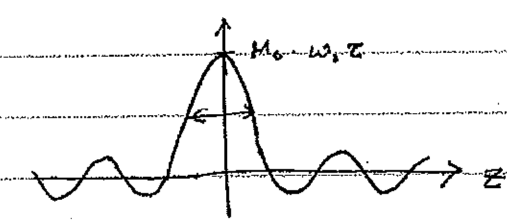



$$- \; M_r(\tau, z) = i\, M_0\, e^{-i\omega(z)\tau/2}\, \mathcal{F}_{1D}\left\{ \omega_1\left(t+\frac{\tau}{2}\right) \right\}\Big|_{f = \frac{-\gamma}{2\pi} G_z \cdot z}$$

$$\omega_1\left(t+\frac{\tau}{2}\right) = \underbrace{\gamma \cdot B_1}_{\omega_1} \cdot \Pi\left(\frac{t}{\tau}\right)$$

$$\mathcal{F}_{1D}\left\{ \Pi\left(\frac{t}{\tau}\right) \right\} = \tau\, \mathrm{sinc}(\tau \cdot f)$$

$$\Rightarrow M_r(\tau, z) = i\, M_0\, \underbrace{e^{-i\omega(z)\tau/2}}_{\text{Canceled out by the refocusing pulse}} \cdot\, \omega_1 \tau\, \mathrm{sinc}\left(\tau \cdot \frac{\gamma}{2\pi} G_z \cdot z\right)$$




$$\Delta z = \frac{1}{\frac{\tau}{2\pi}\gamma \cdot G_z}$$