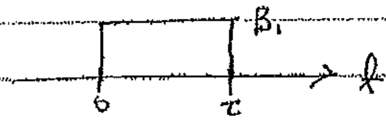
* Small tip-angle example

- consider a rectangular RF pulse (duration of τ)

$$B_1(t) = B_1 \cdot \Pi\left(\frac{t - \tau/2}{\tau}\right)$$

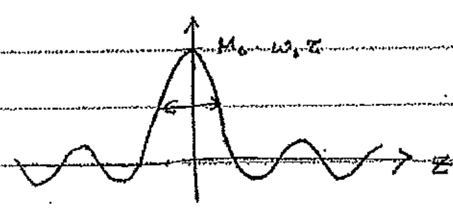


- $$M_r(\tau, z) = i M_0 e^{-i\omega(z)\tau/2} \, \mathcal{F}_{1D}\left\{ \omega_1\left(t + \frac{\tau}{2}\right) \right\} \Big|_{f = -\frac{\gamma}{2\pi} G_z \cdot z}$$

$$\omega_1\left(t + \frac{\tau}{2}\right) = \underbrace{\gamma \cdot B_1}_{\omega_1} \cdot \Pi\left(\frac{t}{\tau}\right)$$

$$\mathcal{F}_{1D}\left\{ \Pi\left(\frac{t}{\tau}\right) \right\} = \tau \, \mathrm{sinc}(\tau \cdot f)$$

$$\Rightarrow M_r(\tau, z) = i \, M_0 \underbrace{e^{-i\omega(z)\tau/2}}_{\text{canceled out by the refocusing pulse}} \cdot \omega_1 \tau \, \mathrm{sinc}\left(\tau \cdot \frac{\gamma}{2\pi} G_z \cdot z\right)$$



$$\Delta z = \frac{1}{\frac{\tau}{2\pi} \gamma G_z}$$

* Full analytical solution

$$\frac{d\vec{M}}{dt} = \begin{bmatrix} 0 & \omega(z) & 0 \\ -\omega(z) & 0 & \omega_1(t) \\ 0 & -\omega_1(t) & 0 \end{bmatrix} \vec{M}$$

$M_r = M_x + i M_y \leftarrow$ phasor representation

$$\frac{dM_r}{dt} = -i\,\omega(z)\,M_r + i\,\omega_1(t)\,M_z$$

$$\frac{dM_z}{dt} = -i\,\omega_1(t)\,M_y$$

Solution $\Longrightarrow$ very difficult $\Longrightarrow$ using approximation

$\sin\theta \simeq \theta$

$\cos\theta \simeq 1$

$M_z \simeq M_0 \simeq$ constant

$$\Rightarrow \frac{dM_z}{dt} = 0$$

$$\frac{dM_r}{dt} = -i\,\omega(z)\,M_r + i\,\omega_1(t)\,M_0$$

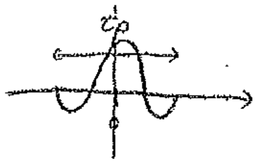
$$\frac{dy}{dx} + p(x)\,y = q(x)$$

$$y = \frac{\int u(x)\,q(x)\,dx}{u(x)}, \quad \text{where} \quad u(x) = \exp\left\{\int p(x)\,dx\right\}$$

$$\frac{d\overset{y}{M_r}}{\underset{x}{dt}} + \underbrace{i\,\omega(z)}_{p(x)}\,\overset{y}{M_r} = \underbrace{i\;\omega_1(t)\,M_0}_{q(x)}$$

$$M_r(t,z) = i\,M_0\,e^{-i\omega(z)\cdot t}\int_0^t \omega_1(\tau)\,e^{-i\omega(z)\cdot\tau}\,d\tau$$

Assume, the RF pulse is symmetric and peaks at $t=0$ such that the pulse ends at $t = \tau_p/2$, τ_p is the length of the RF pulse $\longrightarrow$ let $\tau' = \tau - \tau_p/2$



$$M_r(\tau_p) = i\,M_0\,e^{-\frac{i\omega(z)\tau_p}{2}}\int_{-\tau_p/2}^{\tau_p/2}\omega_1\left(t+\frac{\tau_p}{2}\right)e^{-i\omega(z)t}\,dt$$

assuming $\omega_1\left(t+\frac{\tau_p}{2}\right) = \begin{cases} 0 & \text{for } |t| > \frac{\tau_p}{2} \\ \omega_1\left(t+\frac{\tau_p}{2}\right) & \text{for } |t| < \frac{\tau_p}{2}\end{cases}$

$$\Rightarrow M_r(\tau,z) = i\,M_0\,e^{-i\omega(z)\tau/2}\,\mathcal{F}_{1D}\left\{\omega_1\left(t+\frac{\tau}{2}\right)\right\}\Bigg|_{f=-f(z) = -(\gamma/2\pi)G_z\cdot z}$$

$$* \quad |M_{xy}(\tau, z)| = M_0 \, \mathcal{F}_{1D} \left\{ W_1 \left(t + \frac{\tau}{2} \right) \right\} \Big|_{f = -\frac{\gamma}{2\pi} G_z z}$$