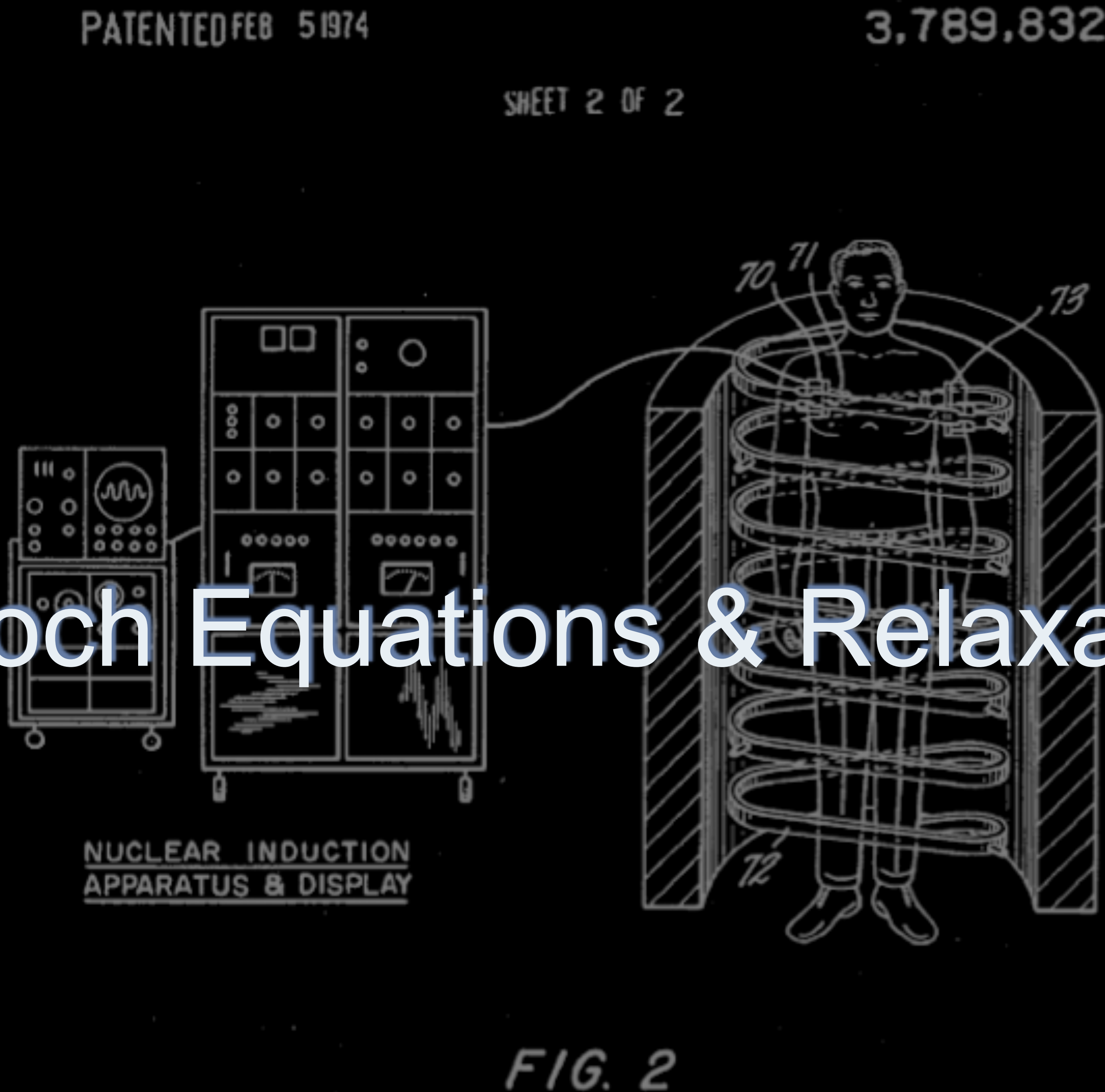
Bloch Equations & Relaxation



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MRI Systems II – B_1

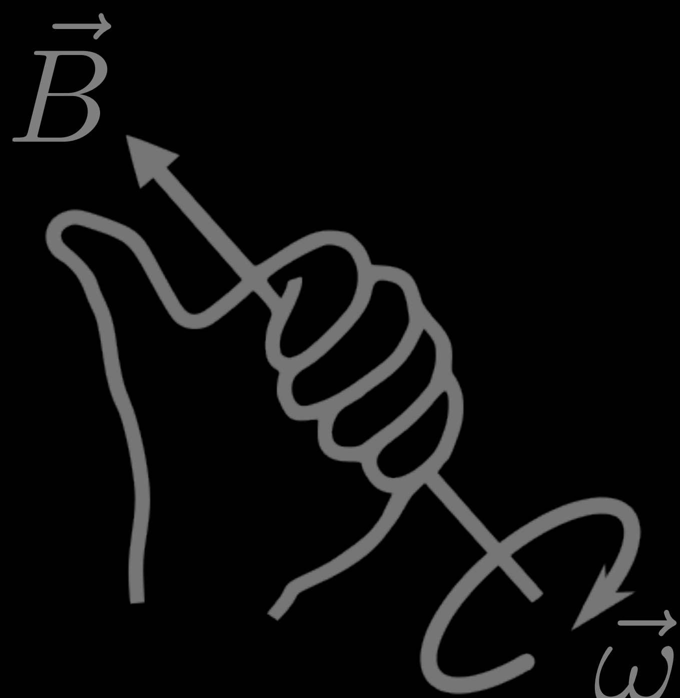
Lecture #3 Learning Objectives

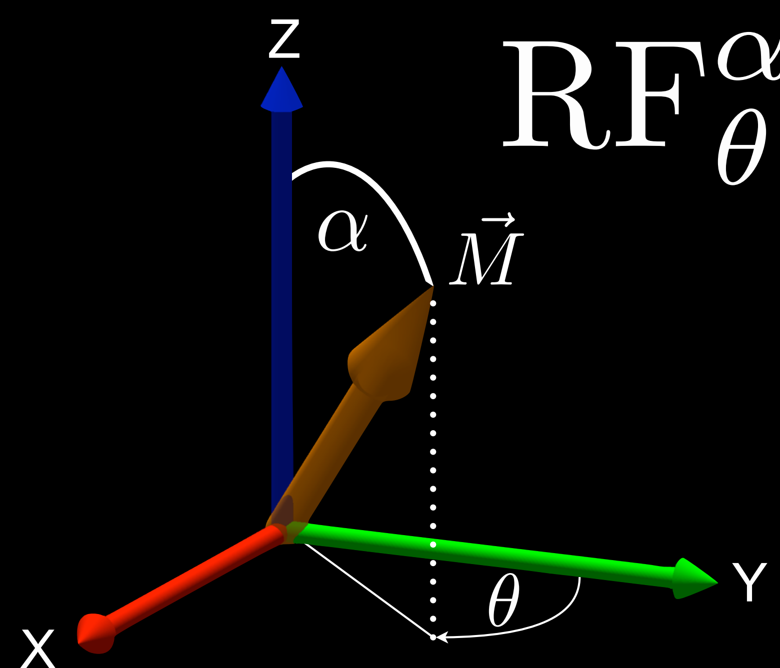
- **Distinguish spin, precession, and nutation.**
- **Appreciate that any B-field acts on the the spin system.**
- **Understand the advantage of a circularly polarized RF B-field.**
- **Differentiate the lab and rotating frames.**
- **Define the equation of motion in the lab and rotating frames.**
- **Know how to compute the flip angle from the B1-envelope function.**
- **Understand how to apply the RF hard pulse matrix operator.**

Mathematics of Hard RF Pulses

Parameters & Rules for RF Pulses

- **RF pulses have a “flip angle” (α)**
  - RF fields induce **left-hand** rotations
    - All B-fields do this for positive γ
- **RF pulses have a “phase” (θ)**
  - Phase of 0° is about the x-axis
  - Phase of 90° is about the y-axis

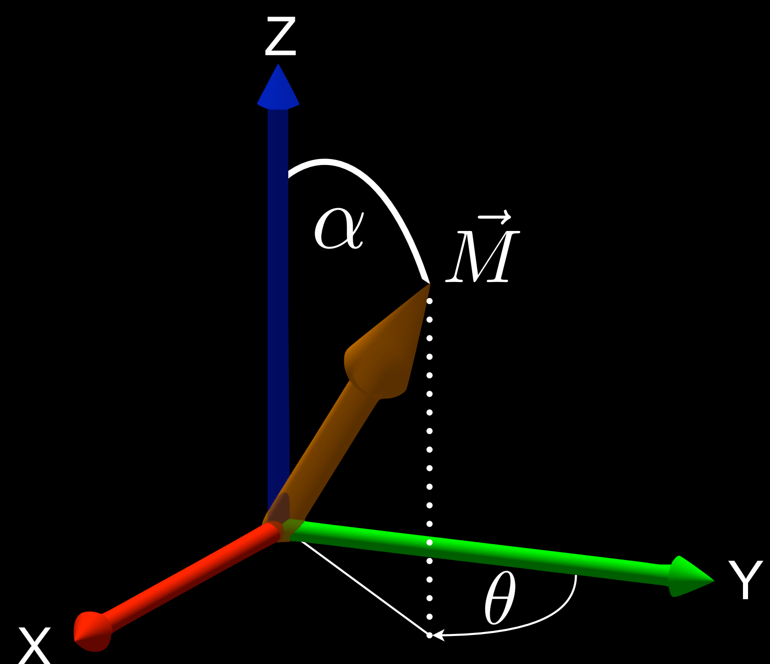




RF Flip Angle

Flip Angle

- **"Amount of rotation of the bulk magnetization vector produced by an RF pulse, with respect to the direction of the static magnetic field."**
  - Liang & Lauterbur, p. 374



$$\omega_1 = \gamma B_1$$

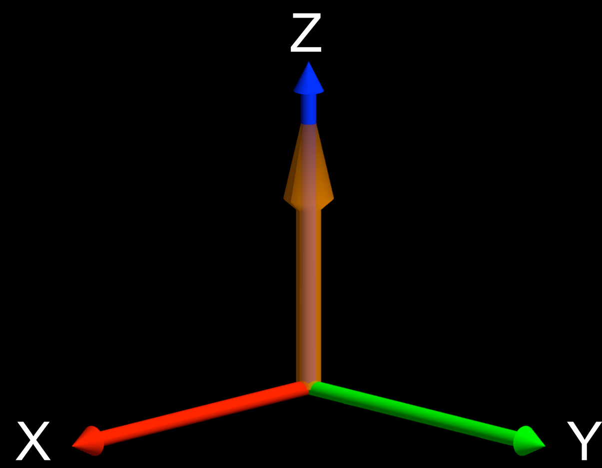
B-fields induce precession!

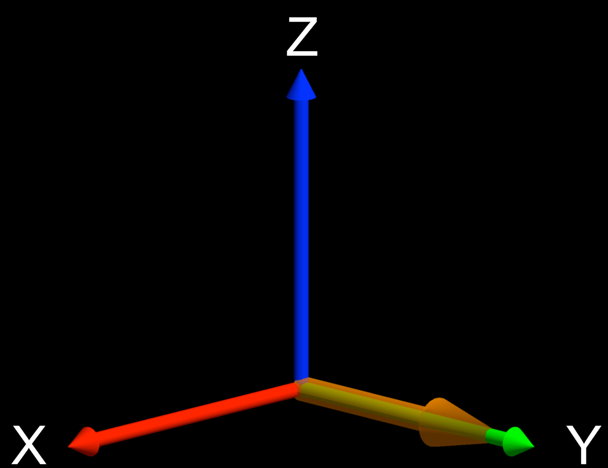
Rules for RF Pulses

$$\mathbf{RF}_{\theta}^{\alpha}$$

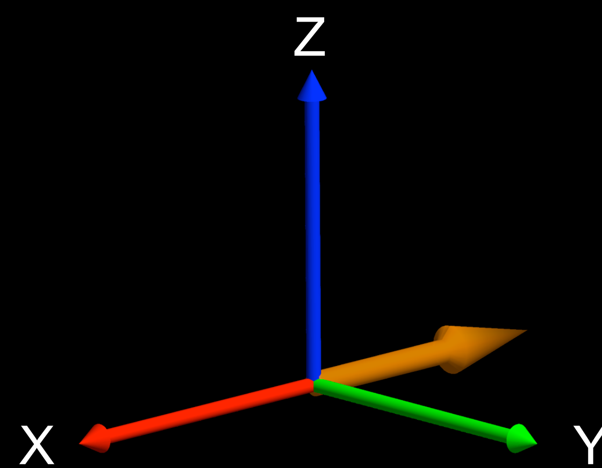
α → Flip Angle

θ → Phase





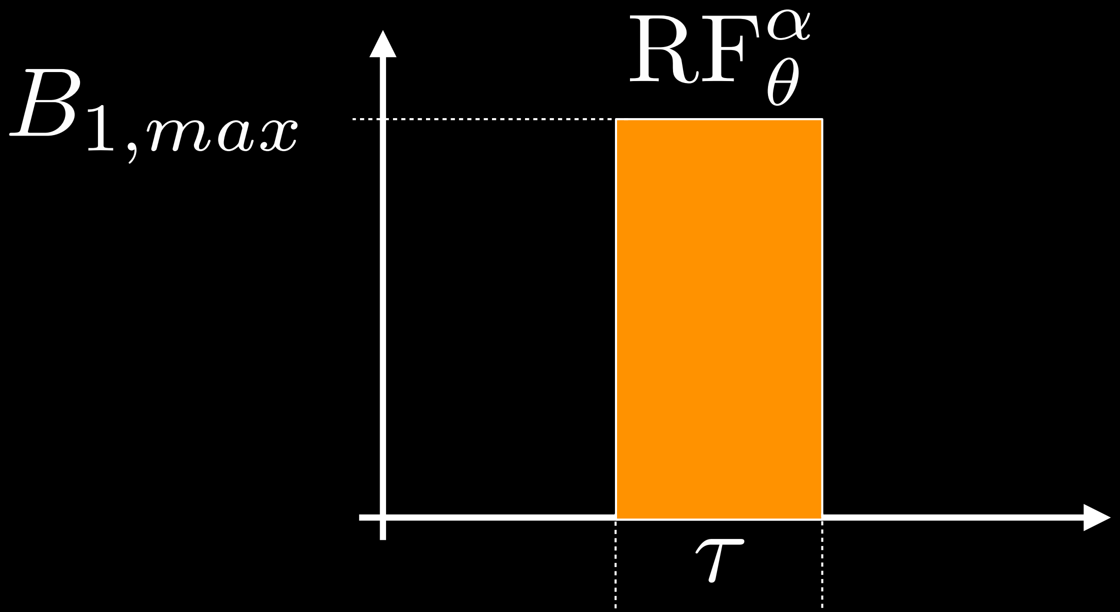
$$\mathbf{RF}_{0^\circ}^{90^\circ}$$



$$\mathbf{RF}_{90^\circ}^{90^\circ}$$

B-fields induce left-handed nutation!

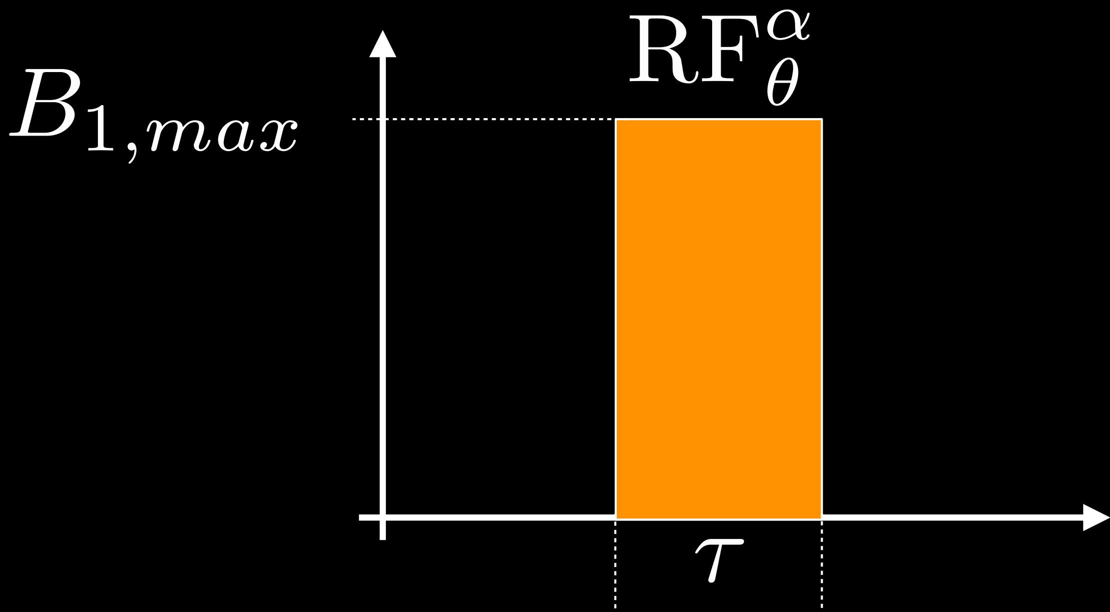
How to determine α?



$$\alpha = \gamma \int_0^{\tau_p} B_1^e(t)\, dt$$

Rules: 1) Specify α
2) Use $B_{1,max}$ if we can
3) Shortest duration pulse

How to determine α?

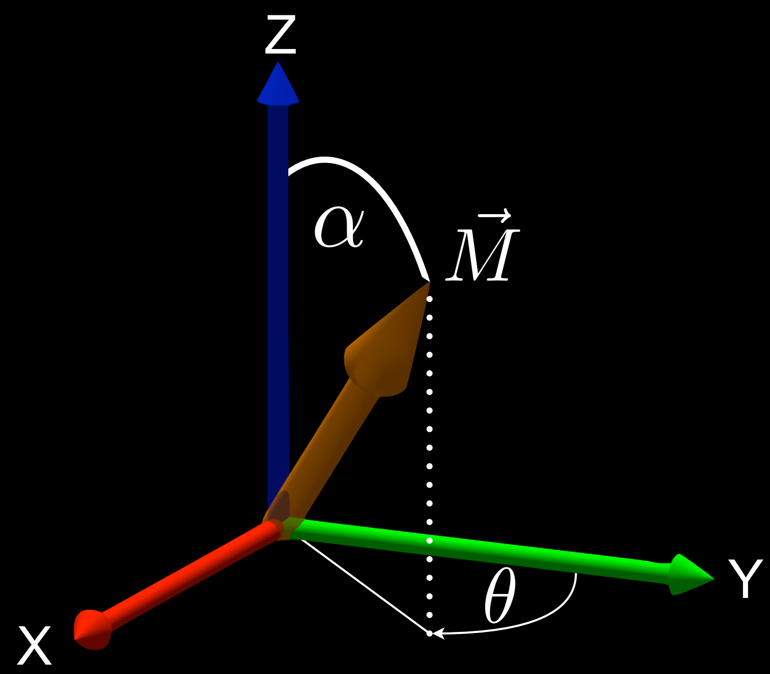


$$\alpha = \gamma \int_0^{\tau_p} B_1^e\left(t\right) dt$$

$$\tau = \frac{\alpha}{\gamma B_{1,max}} = \frac{\pi/2}{2\pi \cdot 42.57Hz/\mu T \cdot 60\mu T} = 0.098ms$$

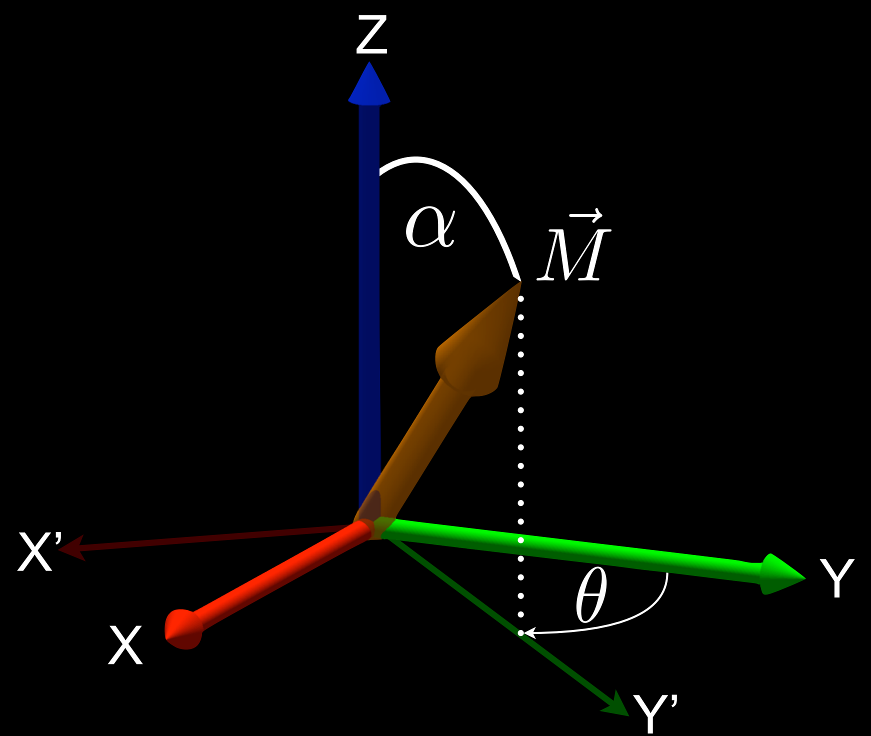
RF Phase

Bulk Magnetization in the Lab Frame



How do we mathematically account for α and θ?

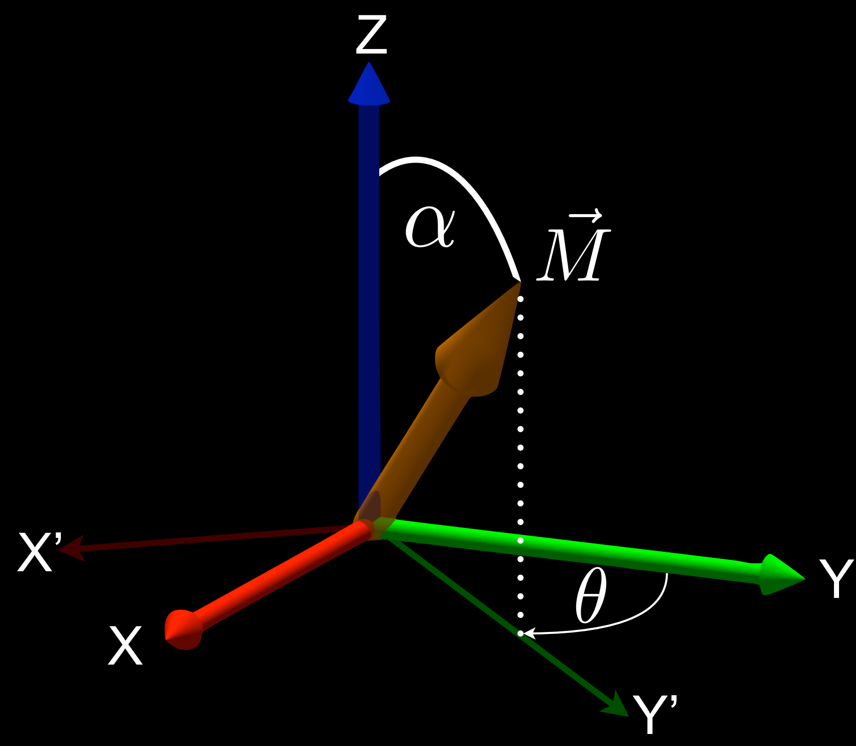
Change of Basis (θ)



$$\mathbf{R}_Z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotate into a coordinate system where ***M*** falls along the y'-axis.

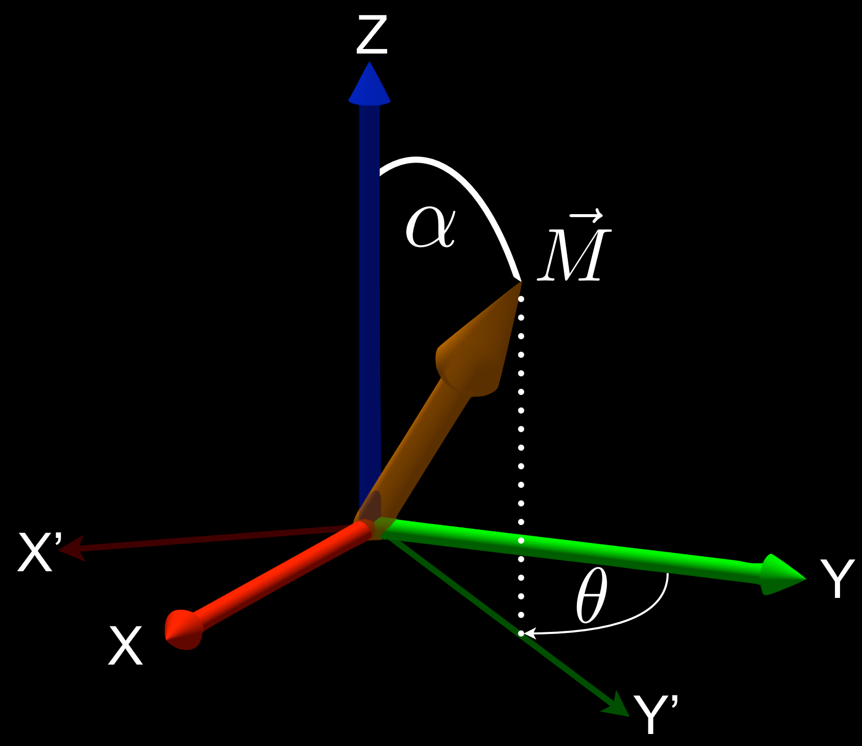
Rotation by Alpha



$$\mathbf{R}_{X'}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

Rotate ***M*** by α about x'-axis.

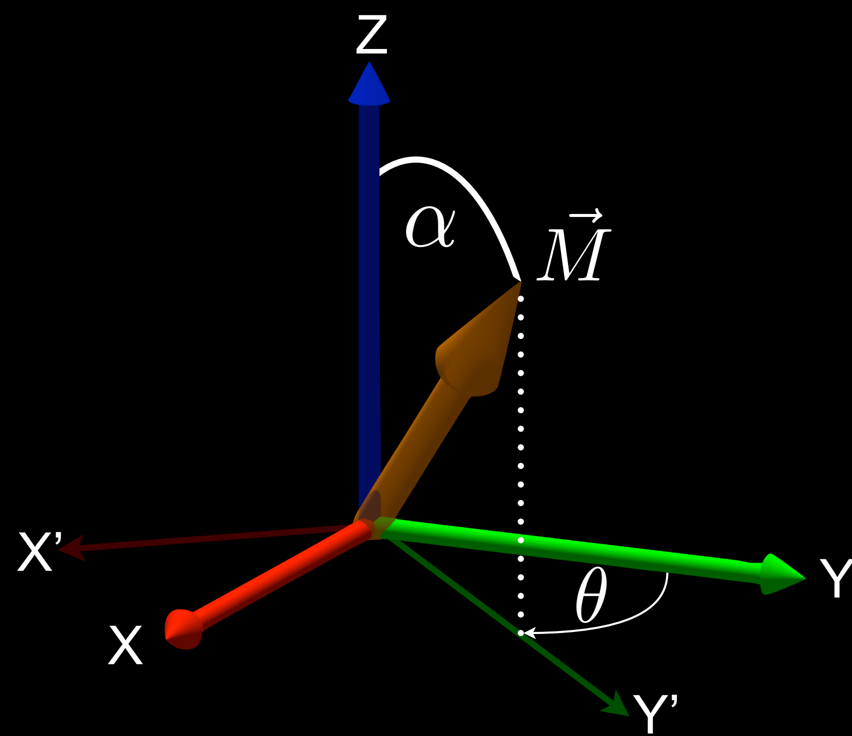
Change of Basis (-θ)



$$\mathbf{R}_Z(-\theta) = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotate back to the lab frame's x-axis and y-axis.

RF Pulse Operator



$$\mathbf{R}_\theta^\alpha = \mathbf{R}_Z(-\theta)\,\mathbf{R}_X(\alpha)\,\mathbf{R}_Z(\theta)$$

$$= \begin{bmatrix} c^2\theta + s^2\theta c\alpha & c\theta s\theta - c\theta s\theta c\alpha & -s\theta s\alpha \\ c\theta s\theta - c\theta s\theta c\alpha & s^2\theta + c^2\theta c\alpha & c\theta s\alpha \\ s\theta s\alpha & -c\theta s\alpha & c\alpha \end{bmatrix}$$

This is the composite matrix operator for a hard RF pulse.

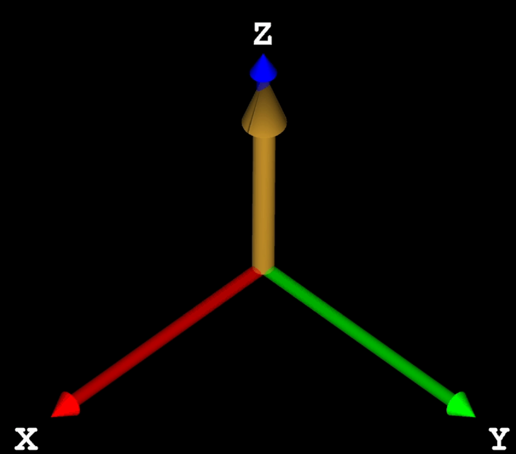
Types of RF Pulses

Types of RF Pulses

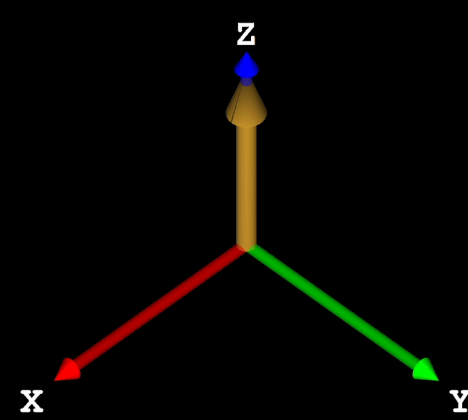
- Excitation Pulses
- Inversion Pulses
- Refocusing Pulses
- Saturation Pulses
- Spectrally Selective Pulses
- Spectral-spatial Pulses
- Adiabatic Pulses

Excitation Pulses

- **Tip M_z into the transverse plane**
- **Typically 200μs to 5ms**
- **Non-uniform across slice thickness**
  - Imperfect slice profile
- **Non-uniform within slice**
  - Termed B_1 inhomogeneity
  - Non-uniform signal intensity across FOV



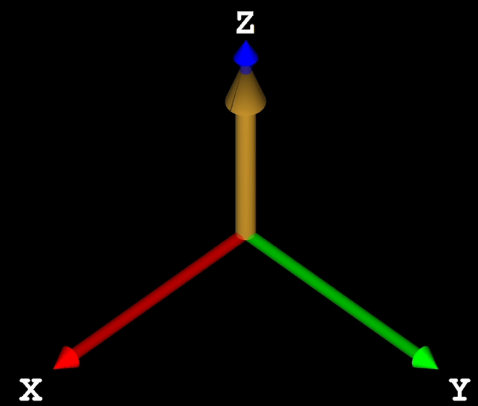
90° Excitation Pulse



Small Flip Angle Pulse

Inversion Pulses

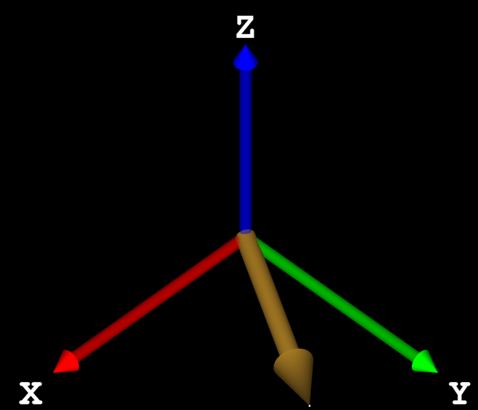
- **Typically, 180° RF Pulse**
  - non-180° that still results in $-M_Z$
- **Invert M_Z to $-M_Z$**
  - Ideally produces no M_{XY}
- **Hard Pulse**
  - Constant RF amplitude
  - Typically non-selective
- **Soft (Amplitude Modulated) Pulse**
  - **Frequency/spatially/spectrally selective**
- **Typically followed by a crusher gradient**



180° Inversion Pulse

Refocusing Pulses

- **Typically, 180° RF Pulse**
  - Provides optimally refocused M_{XY}
  - Largest spin echo signal
- **Refocus spin dephasing due to**
  - imaging gradients
  - local magnetic field inhomogeneity
  - magnetic susceptibility variation
  - chemical shift
- **Typically followed by a crusher gradient**



180° Refocusing Pulse

Lecture #3 Summary - RF Pulses

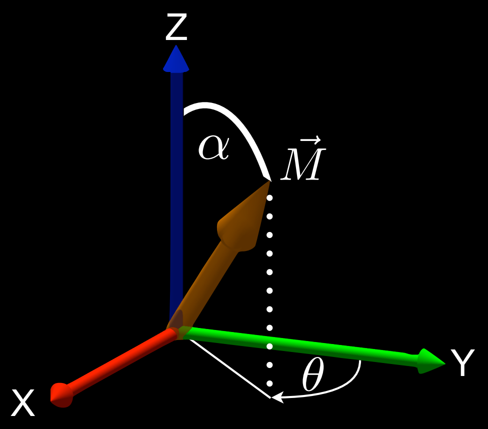
$$\vec{B}_1(t) = \left[\cos(\omega_{RF}t)\hat{i} - \sin(\omega_{RF}t)\hat{j}\right]$$

Circularly Polarized RF Fields

$$\begin{aligned}\mathbf{R}_\theta^\alpha &= \mathbf{R}_Z\left(-\theta\right)\mathbf{R}_X\left(\alpha\right)\mathbf{R}_Z\left(\theta\right)\\ &= \begin{bmatrix} \mathrm{c}^2\theta + \mathrm{s}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & -\mathrm{s}\theta\mathrm{s}\alpha \\ \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & \mathrm{s}^2\theta + \mathrm{c}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\alpha \\ \mathrm{s}\theta\mathrm{s}\alpha & -\mathrm{c}\theta\mathrm{s}\alpha & \mathrm{c}\alpha \end{bmatrix}\end{aligned}$$

RF Pulse Operator

$$\alpha = \gamma\int_0^{\tau_p} B_1^e\left(t\right)dt$$



Choosing the flip angle.

Lecture #3 Summary - Rotating Frame

$$\frac{d\vec{M}_{rot}}{dt} = \vec{M}_{rot} \times \gamma \left(\frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot} \right)$$

Equation of Motion for the Bulk Magnetization in the Rotating Frame Without Relaxation

$$\vec{B}_{eff} \equiv \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot}$$

Definition of the "effective" B-field.

$\frac{\vec{\omega}_{rot}}{\gamma}$ "Fictitious field" that demodulates description of the bulk magnetization.

$\vec{B}_{rot}$ The applied B-field in the Rotating Frame.

$$\frac{d\vec{M}_{rot}}{dt} = \vec{M}_{rot} \times \gamma \vec{B}_{eff}$$

Equation of Motion for the Bulk Magnetization in the Rotating Frame Without Relaxation

Free Precession in the Rotating Frame without Relaxation

$$\vec{B}_{eff} = \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot}$$

$$= \frac{-\gamma B_0 \hat{k}'}{\gamma} + B_0 \hat{k}'$$

$$= 0$$

$$\frac{d\vec{M}_{rot}}{dt} = \vec{M}_{rot} \times \gamma \vec{B}_{eff}$$

$$\frac{d\vec{M}_{rot}}{dt} = \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ M_{x'} & M_{y'} & M_{z'} \\ 0 & 0 & 0 \end{vmatrix}$$

$$\frac{dM_{x'}}{dt} = 0$$

$$\frac{dM_{y'}}{dt} = 0$$

$$\frac{dM_{z'}}{dt} = 0$$

Forced Precession in the Rotating Frame without Relaxation

$$\begin{aligned}\frac{d\vec{M}_{rot}}{dt} &= \vec{M}_{rot} \times \gamma \vec{B}_{eff} \\ &= \vec{M}_{rot} \times \gamma B_1^e(t)\hat{i}' \\ &= \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ \vec{M}_{x'} & \vec{M}_{y'} & \vec{M}_{z'} \\ \gamma B_1^e(t) & 0 & 0 \end{vmatrix}\end{aligned}$$

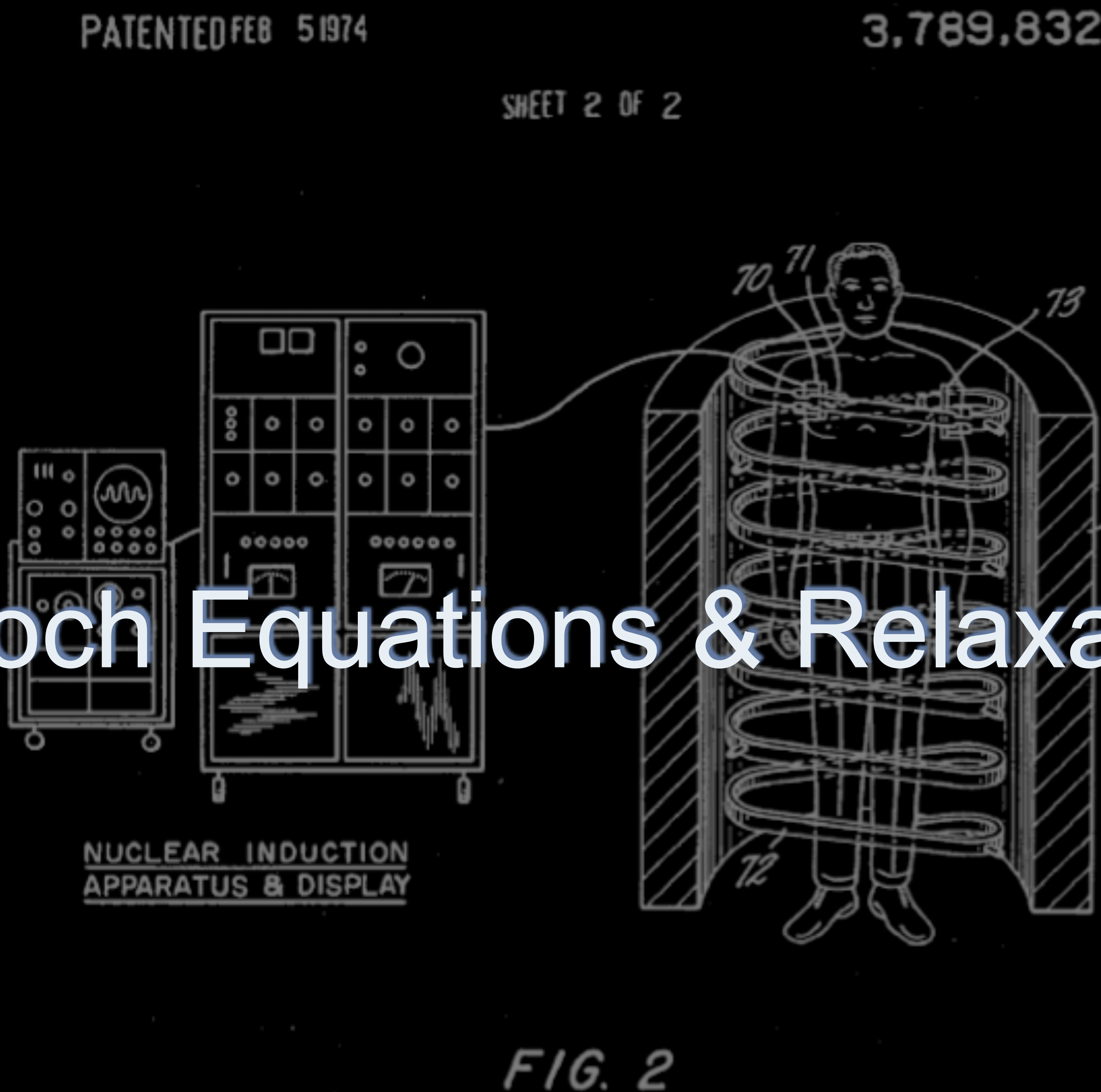
$$\frac{dM_{x'}}{dt} = 0$$

$$\frac{dM_{y'}}{dt} = \gamma B_1^e(t) M_{z'}$$

$$\frac{dM_{z'}}{dt} = -\gamma B_1^e(t) M_{y'}$$

To The Board…

Bloch Equations & Relaxation



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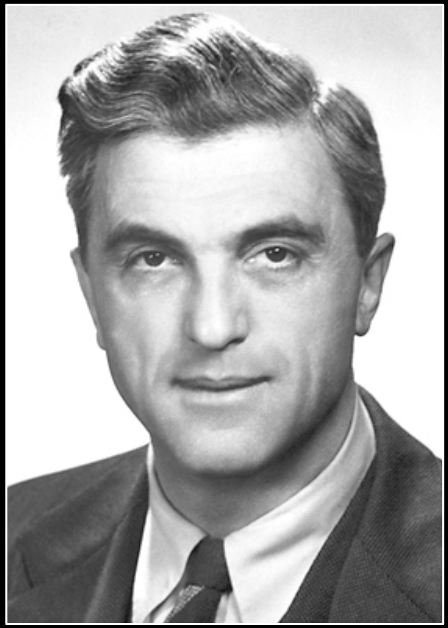
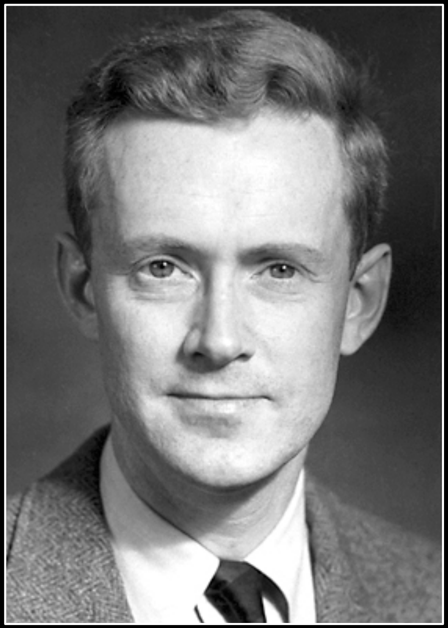
1952 Nobel Prize in Physics

“for their development of new methods for nuclear magnetic precision measurements and discoveries in connection therewith“



Felix Bloch
b. 23 Oct 1905
d. 10 Sep 1983



Edward Purcell
b. 30 Sep 1912
d. 07 Mar 1997

Bloch Equations with Relaxation

$$\frac{d\vec{\mathrm{M}}}{dt} = \vec{\mathrm{M}} \times \gamma\vec{\mathrm{B}} - \frac{M_x\hat{\mathrm{i}} + M_y\hat{\mathrm{j}}}{T_2} - \frac{(M_z - M_0)\,\hat{\mathrm{k}}}{T_1}$$

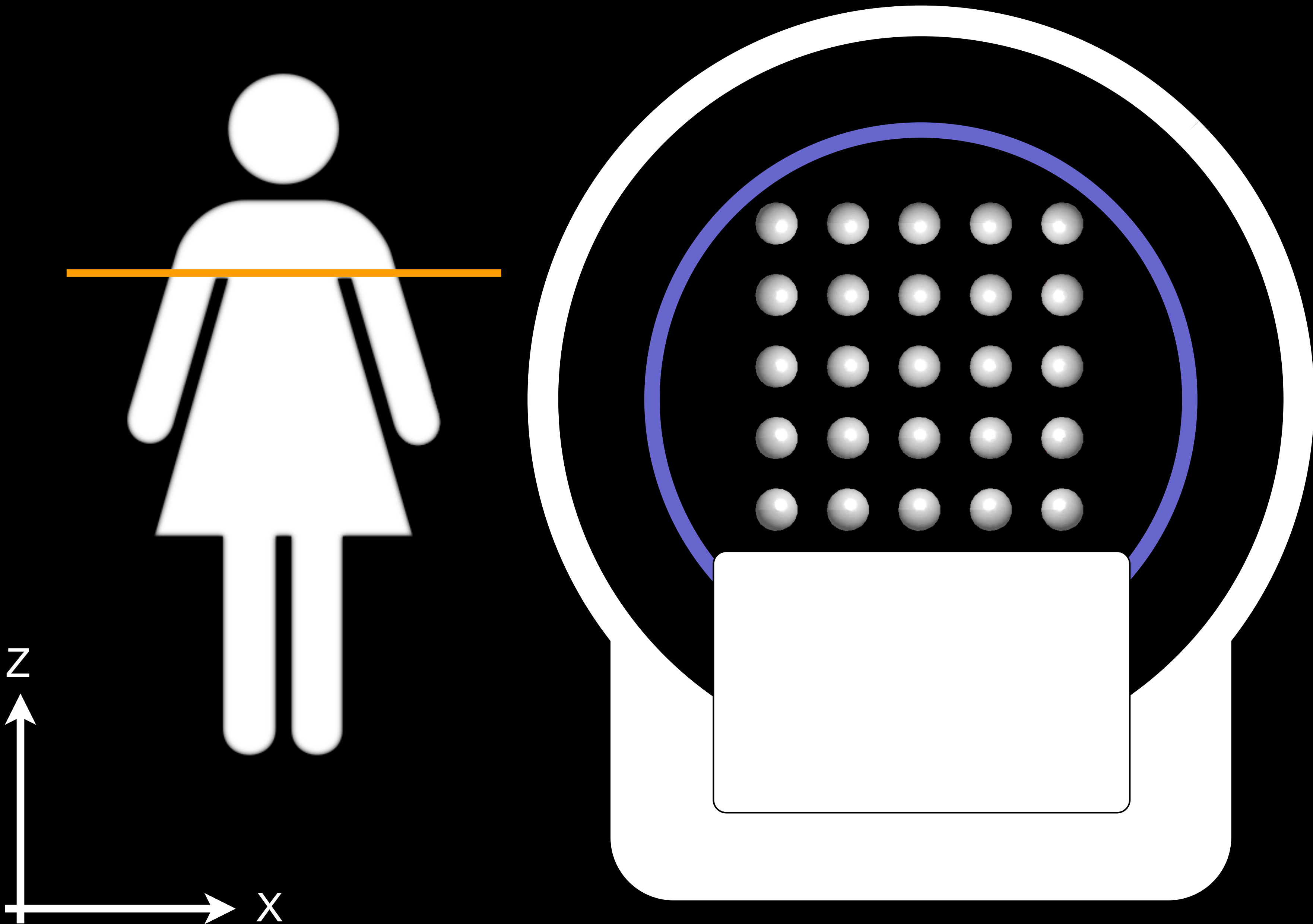
- **Differential Equation**
  - Ordinary, Coupled, Non-linear
- **No analytic solution, in general.**
  - Analytic solutions for simple cases.
  - Numerical solutions for all cases.
- **Phenomenological**
  - Exponential behavior is an approximation.

Bloch Equations - Lab Frame

$$\frac{d\vec{\mathrm{M}}}{dt} = \underbrace{\vec{\mathrm{M}} \times \gamma \vec{\mathrm{B}}}_{\text{Precession}} - \underbrace{\frac{M_x \hat{\mathrm{i}} + M_y \hat{\mathrm{j}}}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_z - M_0)\,\hat{\mathrm{k}}}{T_1}}_{\text{Longitudinal Relaxation}}$$

- Precession
  - Magnitude of M unchanged
  - Phase (rotation) of M changes due to B
- Relaxation
  - T_1 changes are slow O(100ms)
  - T_2 changes are fast O(10ms)
  - Magnitude of M can be ZERO
- Diffusion
  - Spins are thermodynamically driven to exchange positions.
  - Bloch-Torrey Equations

Excitation and Relaxation



The magnetization relaxes after excitation (forced precession).

Bloch Equations – Rotating Frame

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \underbrace{\gamma \vec{M}_{rot} \times \vec{B}_{eff}}_{\text{“Precession”}} - \underbrace{\frac{M_{x'}\vec{i}' + M_{y'}\vec{j}'}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_{z'} - M_0)\vec{k}'}{T_1}}_{\text{Longitudinal Relaxation}}$$

$$\vec{B}_{eff} \equiv \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot}$$

Effective B-field that *M* experiences in the rotating frame.

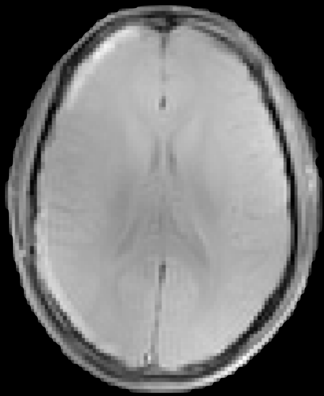
Fictitious field that demodulates the apparent effect of B_0

Applied B-field in the rotating frame.

T_1 Relaxation

T1 and T2 Values

Tissue	T1 [ms]	T2 [ms]
gray matter	925	100
white matter	790	92
muscle	875	47
fat	260	85
kidney	650	58
liver	500	43
CSF	2400	180



TI=25ms
TE=12ms

TI=200ms
TE=12ms

TI=500ms
TE=12ms

TI=1000ms
TE=12ms

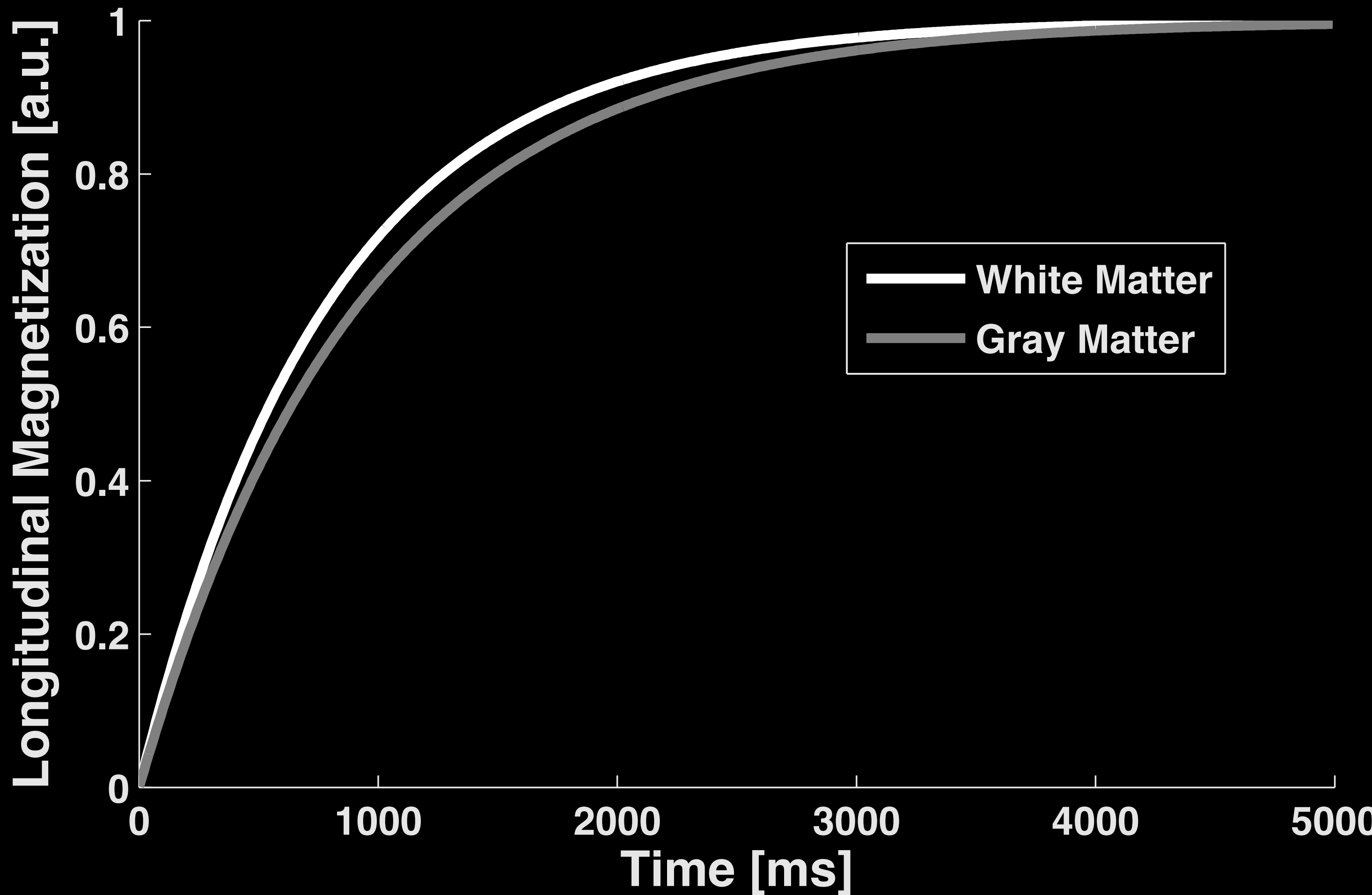
Each tissue as “unique” relaxation properties.

T_1 Relaxation

- Longitudinal or spin-lattice relaxation
- Typically 100s to 1000s of ms
- T_1 increases with increasing B_0
- T_1 decreases with contrast agents
- Short T_1s are bright on T_1-weighted image

T_1 Relaxation

Tissue	T_1 [ms]	T_2 [ms]
gray matter	925	100
white matter	790	92



T_1 Relaxation

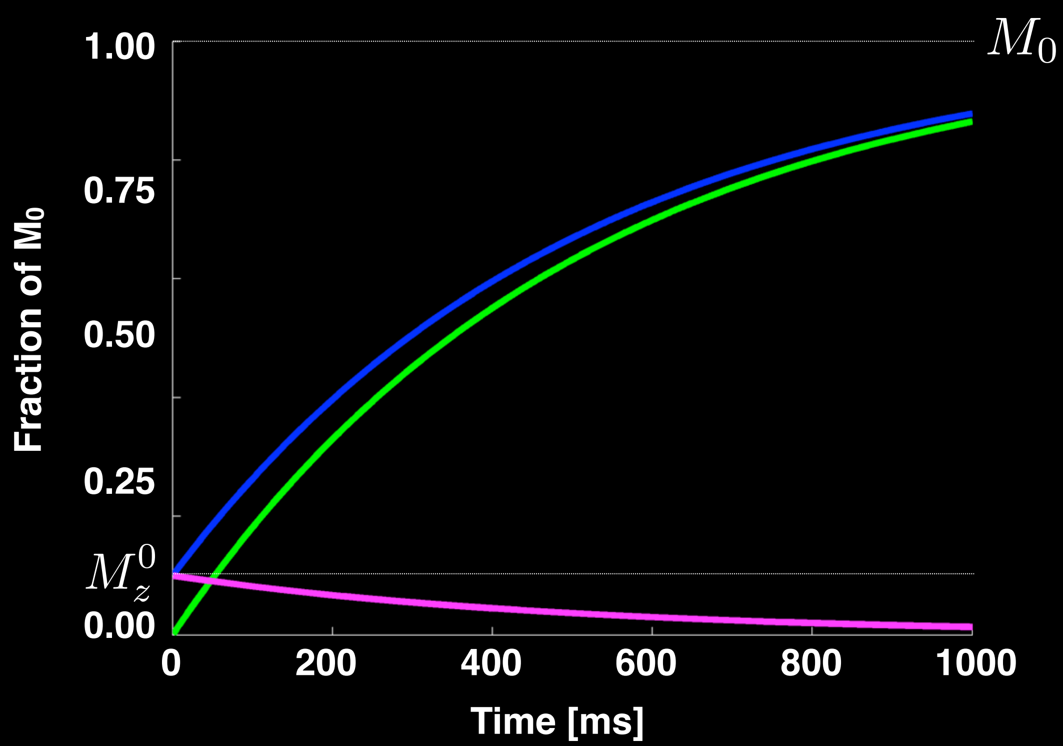
Free Precession in the Lab ***or*** Rotating Frame with Relaxation

$$M_z(t) = \underbrace{M_z^0 e^{-\frac{t}{T_1}}}_{} + \underbrace{M_0\left(1 - e^{-\frac{t}{T_1}}\right)}_{}$$

Net Magnetization — $M_z(t)$

Prepared Magnetization Decays (M_z^0) — $M_z^0 e^{-\frac{t}{T_1}}$

Return to Thermal Equilibrium (M_0) — $M_0\left(1 - e^{-\frac{t}{T_1}}\right)$



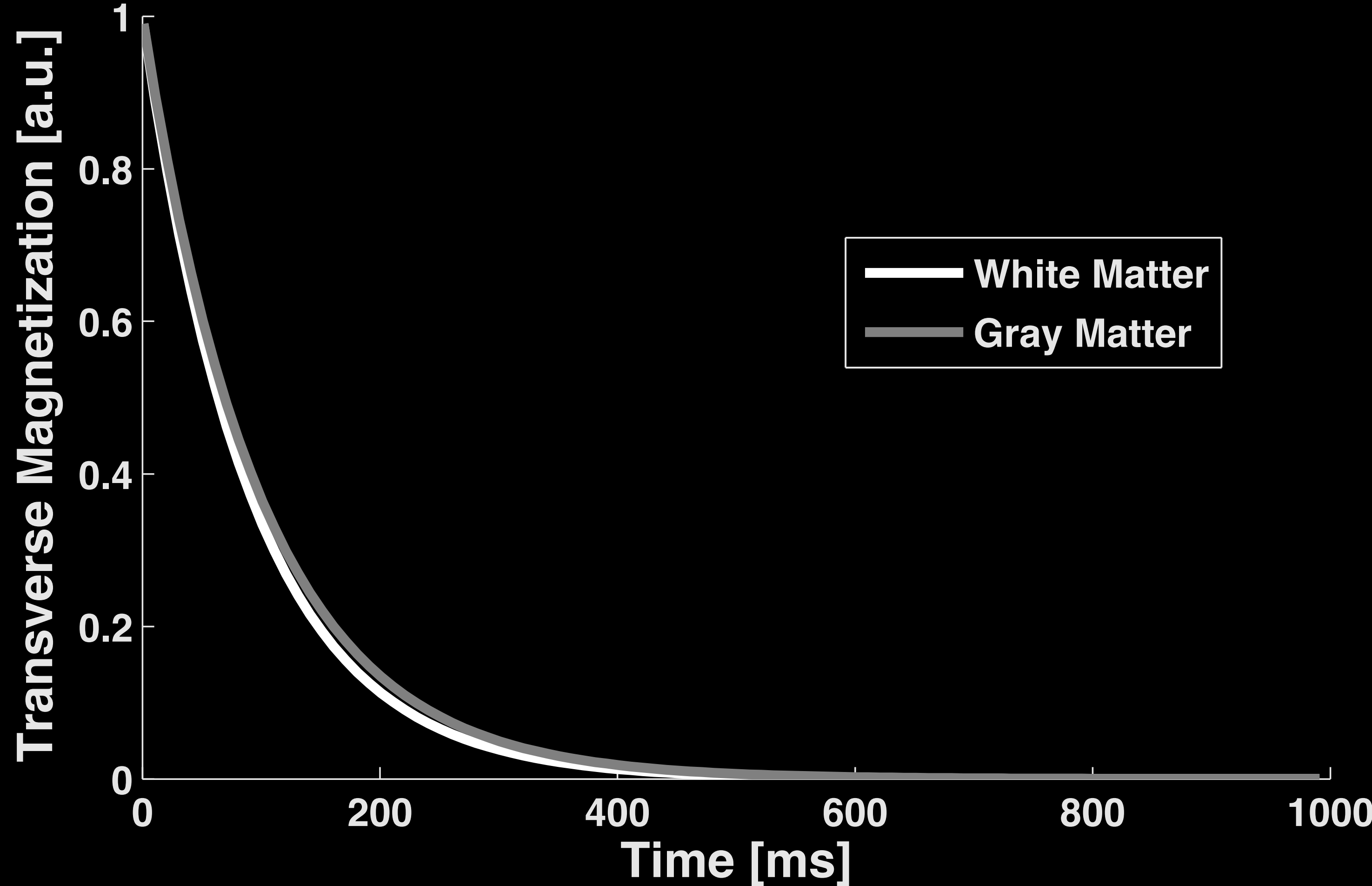
T_2 Relaxation

T_2 Relaxation

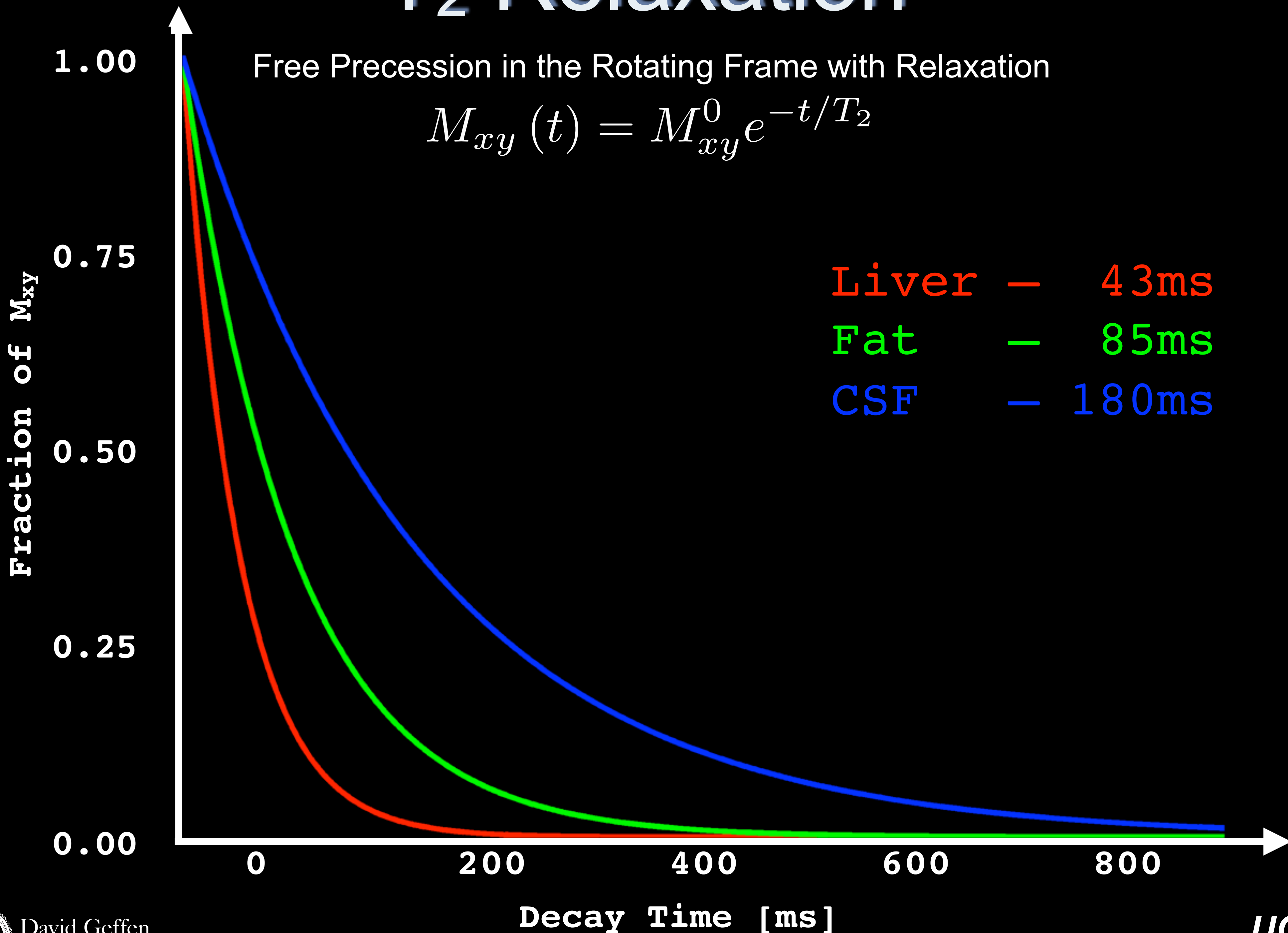
- Transverse or spin-spin relaxation
  - Molecular interaction causes spin dephasing
- T_2 typically 10s to 100s of ms
- T_2 relatively independent of B_0
- T_2 always < T_1
- T_2 decreases with contrast agents
- Long T_2 is bright on T_2 weighted image

T_2 Relaxation

Tissue	T_1 [ms]	T_2 [ms]
gray matter	925	100
white matter	790	92



T_2 Relaxation



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Matlab

Bloch Equation Simulations

Rotating Frame Bloch Equations (Free Precession)

$$\frac{d\vec{\mathrm{M}}}{dt} = -\frac{\mathrm{M_x}\hat{i} + \mathrm{M_y}\hat{j}}{T_2} - \frac{(\mathrm{M_z} - \mathrm{M_0})\,\hat{k}}{T_1}$$

Rotating Frame Bloch Equations (Free Precession)

$$\frac{d\vec{\mathrm{M}}}{dt} = -\frac{\mathrm{M_x}\hat{i} + \mathrm{M_y}\hat{j}}{T_2} - \frac{(\mathrm{M_z} - \mathrm{M_0})\,\hat{k}}{T_1}$$

⇓

$$\begin{bmatrix} \frac{dM_x}{dt} \\ \frac{dM_y}{dt} \\ \frac{dM_z}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2} & 0 & 0 \\ 0 & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{M_0}{T_1} \end{bmatrix}$$

Rotating Frame Bloch Equations

$$\frac{d\vec{\mathrm{M}}}{dt} = -\frac{\mathrm{M_x}\hat{i} + \mathrm{M_y}\hat{j}}{T_2} - \frac{(\mathrm{M_z} - \mathrm{M_0})\,\hat{k}}{T_1}$$

$$\begin{bmatrix} \frac{dM_x}{dt} \\ \frac{dM_y}{dt} \\ \frac{dM_z}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2} & 0 & 0 \\ 0 & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{M_0}{T_1} \end{bmatrix}$$

$$\frac{d\vec{\mathrm{M}}}{dt} = \alpha\vec{\mathrm{M}} + \beta$$

An *affine transformation* between two vector spaces consists of a translation followed by a linear transformation.

Why Homogenous Coordinates?

Homogenous coordinates allow us to transform an affine (non-linear) equation in 3D to a linear equation in 4D.

Affine | **Linear**

$$\frac{d\vec{\mathrm{M}}}{dt} = \alpha\vec{\mathrm{M}} + \beta \quad \longleftrightarrow \quad \frac{d\vec{\mathrm{M}}_H}{dt} = \mathbf{T}_H\vec{\mathrm{M}}_H$$

Now we can use the machinery of linear algebra for writing out the Bloch Equation mechanics.

Homogenous Coordinate Expressions

Cartesian Coordinates

$$\vec{\mathrm{M}} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

Homogeneous Coordinates

$$\vec{\mathrm{M}}_H = \begin{bmatrix} M_x \\ M_y \\ M_z \\ 1 \end{bmatrix}$$

$$\xrightarrow{Augment}$$

$$\xleftarrow[Reduce]{}$$

$$\mathbf{T} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

$$\mathbf{T}_H = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} & T_{xt} \\ T_{yx} & T_{yy} & T_{yz} & T_{yt} \\ T_{zx} & T_{zy} & T_{zz} & T_{zt} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotating Frame Bloch Equations (Free Precession)

$$\frac{d\vec{\mathrm{M}}}{dt} = -\frac{\mathrm{M_x}\hat{i} + \mathrm{M_y}\hat{j}}{T_2} - \frac{(\mathrm{M_z} - \mathrm{M_0})\,\hat{k}}{T_1}$$

$$\begin{bmatrix} \frac{dM_x}{dt} \\ \frac{dM_y}{dt} \\ \frac{dM_z}{dt} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2} & 0 & 0 & 0 \\ 0 & -\frac{1}{T_2} & 0 & 0 \\ 0 & 0 & -\frac{1}{T_1} & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{M_0}{T_1} \\ 1 \end{bmatrix}$$

$$\frac{d\vec{\mathrm{M}}_H}{dt} = \mathbf{T}_H \vec{\mathrm{M}}_H$$

Advantages/Disadvantages

+ 1:1 Correlation with pulse diagram
+ Simple to implement (Matlab!)
+ Not *ad hoc*
+ Provides understanding in complex systems

- Masks understanding in simple systems
- Reduction to algebraic expression is cumbersome
- Discrete (not continuous)
- Perfect simulations are very difficult
    - Must consider assumptions
- Image Prep vs. Imaging

B_0 Fields

Bulk Magnetization - Precession

$$\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos\gamma B_0 t & \sin\gamma B_0 t & 0 \\ -\sin\gamma B_0 t & \cos\gamma B_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x^0 \\ M_y^0 \\ M_z^0 \end{bmatrix}$$

$$\vec{M}(t) = \mathbf{R}_z(\gamma B_0 t)\vec{M}^0$$

Bulk Magnetization - Precession

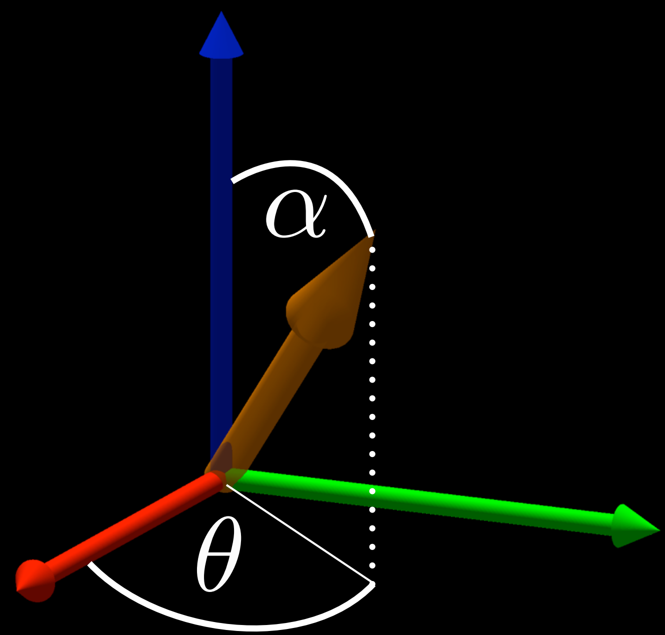
$$B_{0,H} = \begin{bmatrix} \cos\gamma Bt & \sin\gamma Bt & 0 & 0 \\ -\sin\gamma Bt & \cos\gamma Bt & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} M_x(0-) \\ M_y(0-) \\ M_z(0-) \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\gamma Bt & \sin\gamma Bt & 0 & 0 \\ -\sin\gamma Bt & \cos\gamma Bt & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0+) \\ M_y(0+) \\ M_z(0+) \\ 1 \end{bmatrix}$$

Homogeneous coordinate expression for precession.

RF Pulses

RF Pulse Operator



$$\mathbf{R}_\theta^\alpha = \mathbf{R}_Z(-\theta)\,\mathbf{R}_X(\alpha)\,\mathbf{R}_Z(\theta)$$

$$= \begin{bmatrix} \mathrm{c}^2\theta + \mathrm{s}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & -\mathrm{s}\theta\mathrm{s}\alpha \\ \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & \mathrm{s}^2\theta + \mathrm{c}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\alpha \\ \mathrm{s}\theta\mathrm{s}\alpha & -\mathrm{c}\theta\mathrm{s}\alpha & \mathrm{c}\alpha \end{bmatrix}$$

$$\vec{\mathrm{M}}(0_+) = \mathrm{RF}_\theta^\alpha \vec{\mathrm{M}}(0_-)$$

RF Pulse Homogeneous Operator

$$\mathbf{RF}^{\alpha}_{\theta,H} = \begin{bmatrix} \mathrm{c}^2\theta + \mathrm{s}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & -\mathrm{s}\theta\mathrm{s}\alpha & 0 \\ \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & \mathrm{s}^2\theta + \mathrm{c}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\alpha & 0 \\ \mathrm{s}\theta\mathrm{s}\alpha & -\mathrm{c}\theta\mathrm{s}\alpha & \mathrm{c}\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

RF Pulse Homogeneous Operator

$$\mathbf{RF}^{\alpha}_{\theta,H} = \begin{bmatrix} \mathrm{c}^2\theta + \mathrm{s}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & -\mathrm{s}\theta\mathrm{s}\alpha & 0 \\ \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & \mathrm{s}^2\theta + \mathrm{c}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\alpha & 0 \\ \mathrm{s}\theta\mathrm{s}\alpha & -\mathrm{c}\theta\mathrm{s}\alpha & \mathrm{c}\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\mathrm{M}}^{+}_{H} = \mathrm{RF}^{\alpha}_{\theta,H}\vec{\mathrm{M}}^{-}_{H}$$

RF Pulse Homogeneous Operator

$$\mathbf{RF}^{\alpha}_{\theta,H} = \begin{bmatrix} \mathrm{c}^2\theta + \mathrm{s}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & -\mathrm{s}\theta\mathrm{s}\alpha & 0 \\ \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & \mathrm{s}^2\theta + \mathrm{c}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\alpha & 0 \\ \mathrm{s}\theta\mathrm{s}\alpha & -\mathrm{c}\theta\mathrm{s}\alpha & \mathrm{c}\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\mathrm{M}}^{+}_{H} = \mathrm{RF}^{\alpha}_{\theta,H}\vec{\mathrm{M}}^{-}_{H}$$

$$\begin{bmatrix} M_x^+ \\ M_y^+ \\ M_z^+ \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}^2\theta + \mathrm{s}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & -\mathrm{s}\theta\mathrm{s}\alpha & 0 \\ \mathrm{c}\theta\mathrm{s}\theta - \mathrm{c}\theta\mathrm{s}\theta\mathrm{c}\alpha & \mathrm{s}^2\theta + \mathrm{c}^2\theta\mathrm{c}\alpha & \mathrm{c}\theta\mathrm{s}\alpha & 0 \\ \mathrm{s}\theta\mathrm{s}\alpha & -\mathrm{c}\theta\mathrm{s}\alpha & \mathrm{c}\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x^- \\ M_y^- \\ M_z^- \\ 1 \end{bmatrix}$$

Relaxation

Relaxation Operator

$$\begin{bmatrix} M_x^+ \\ M_y^+ \\ M_z^+ \end{bmatrix} = \begin{bmatrix} e^{-\frac{t}{T_2}} & & \\ & e^{-\frac{t}{T_2}} & \\ & & e^{-\frac{t}{T_1}} \end{bmatrix} \begin{bmatrix} M_x^- \\ M_y^- \\ M_z^- \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_0\left(1 - e^{-\frac{t}{T_1}}\right) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-\frac{t}{T_2}} & & & \\ & e^{-\frac{t}{T_2}} & & \\ & & e^{-\frac{t}{T_1}} & M_0\left(1 - e^{-\frac{t}{T_1}}\right) \\ & & & 1 \end{bmatrix} \begin{bmatrix} M_x^- \\ M_y^- \\ M_z^- \\ 1 \end{bmatrix}$$

Relaxation Operator

$$\mathbf{E}\left(T_1, T_2, t, \mathrm{M}_0\right) = \begin{bmatrix} E_2 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_1 & \mathrm{M}_0\left(1 - E_1\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = e^{-t/T_1} \qquad E_2 = e^{-t/T_2}$$

$$\vec{\mathrm{M}}^{+} = \mathrm{E}\left(T_1, T_2, t, \mathrm{M}_0\right)\vec{\mathrm{M}}^{-}$$

*In general we drop the sub-scripted *H*

B_0, RF Pulse, & Relaxation Operators

$$\vec{\mathrm{M}}^{+} = B_{0,H}\vec{\mathrm{M}}^{-}$$

$$\vec{\mathrm{M}}^{+} = \mathrm{RF}_{\theta}^{\alpha}\vec{\mathrm{M}}^{-}$$

$$\vec{\mathrm{M}}^{+} = \mathrm{E}\left(T_1, T_2, t, \mathrm{M}_0\right)\vec{\mathrm{M}}^{-}$$

Matlab Example - B_0

```
% This function returns the 4x4 homogenous coordinate expression for
% precession for a particular gyromagnetic ratio (gamma), external
% field (B0), and time step (dt).
%
% SYNTAX:  dB0=PAM_B0_op(gamma,B0,dt)
%
% INPUTS:  gamma - Gyromagnetic ratio [Hz/T]
%          B0    - Main magnetic field [T]
%          dt    - Time step or vector [s]
%
% OUTPUTS: dB0   - Precessional operator [4x4]
%
% DBE@UCLA 01.21.2015

function dB0=PAM_B0_op(gamma,B0,dt)

if nargin==0
  gamma=42.57e6;          % Gyromagnetic ratio for 1H
  B0=1.5;                 % Typical B0 field strength
  dt=ones(1,100)*1e-6;    % 100 1μs time steps
end

dB0=zeros(4,4,numel(dt));  % Initialize the array

for n=1:numel(dt)
  dw=2*pi*gamma*B0*dt(n);  % Incremental precession (rotation angle)

  % Precessional Operator (left handed)
  dB0(:,:,n)=[ cos(dw)  sin(dw)   0  0;
              -sin(dw)  cos(dw)   0  0;
                0         0       1  0;
                0         0       0  1];
end
return
```

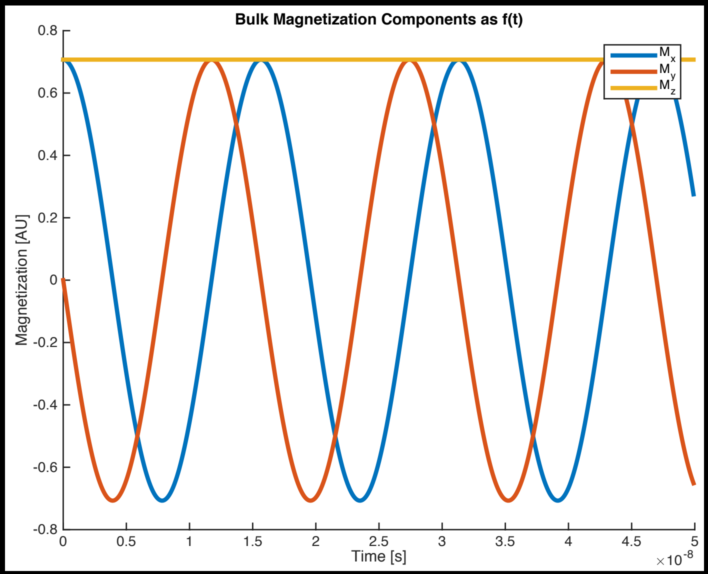
$$\begin{bmatrix} \cos\gamma B_0 t & \sin\gamma B_0 t & 0 \\ -\sin\gamma B_0 t & \cos\gamma B_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matlab Example - Free Precession

```
%% Filename: PAM_Lec02_B0_Free_Precession.m
%
% Demonstrate the precession of the bulk magnetization vector.
%
% DBE@UCLA 2015.01.06

%% Define some constants
gamma=42.57e6;           % Gyromagnetic ratio for 1H [MHz/T]
B0=1.5;                  % B0 magnetic field strength [T]
dt=0.01e-8;              % Time step [s]
nt=500;                  % Number of time points to simulate
t=(0:nt-1)*0.01e-8;      % Time vector [s]

M0=[sqrt(2)/2 0 sqrt(2)/2 1]';  % Initial condition (I.C.)

M=zeros(4,nt);           % Initialize the magnetization array
M(:,1)=M0;               % Define the first time point as the I.C.

%% Simulate precession of the bulk magnetization vector
dB0=PAM_B0_op(gamma,B0,dt); % Calculate the homogenous coordinate transform

for n=2:nt
  M(:,n)=dB0*M(:,n-1);
end
```

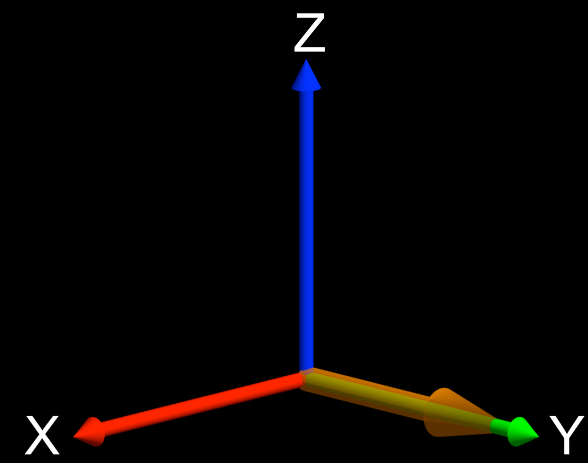
$$\vec{M}(t) = \mathrm{R}_z(\gamma B_0 t)\vec{M}^0$$

```
%% Plot the results
figure; hold on;
  p(1)=plot(t,M(1,:));       % Plot the Mx component
  p(2)=plot(t,M(2,:));       % Plot the My component
  p(3)=plot(t,M(3,:));       % Plot the Mz component
    set(p,'LineWidth',3);    % Increase plot thickness
  ylabel('Magnetization [AU]');
  xlabel('Time [s]');
  legend('M_x','M_y','M_z');
  title('Bulk Magnetization Components as f(t)');
```



Hard RF Pulses

```
function dB1=PAM_B1_op(gamma,B1,dt,theta)

% Define the incremental flip angle in time dt
alpha=2*pi*gamma*B1*dt;

% Change of basis
R_theta=[ cos(theta)  sin(theta) 0 0;
         -sin(theta)  cos(theta) 0 0;
          0           0          1 0;
          0           0          0 1];

% Flip angle rotation
R_alpha=[1  0           0          0;
         0  cos(alpha)  sin(alpha) 0;
         0 -sin(alpha)  cos(alpha) 0;
         0  0           0          1];

% Homogeneous expression for RF MATRIX
dB1=R_theta.'*R_alpha*R_theta;

return
```



$$\mathrm{R}_{0^\circ}^{90^\circ}$$

$$\mathbf{R}_{0^\circ}^{90^\circ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

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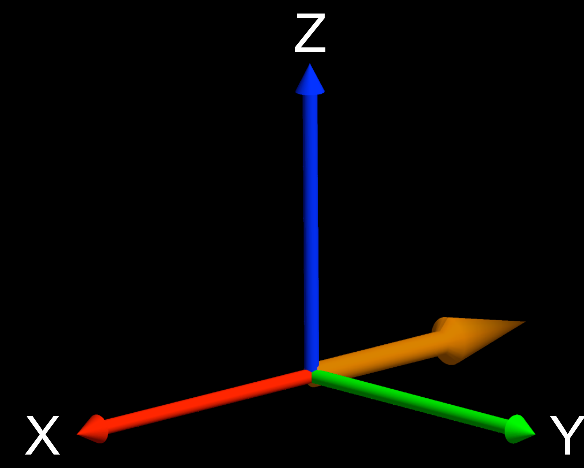
UCLA Radiology

Hard RF Pulses

```
function dB1=PAM_B1_op(gamma,B1,dt,theta)

% Define the incremental flip angle in time dt
alpha=2*pi*gamma*B1*dt;

% Change of basis
R_theta=[ cos(theta)  sin(theta) 0 0;
         -sin(theta)  cos(theta) 0 0;
          0           0          1 0;
          0           0          0 1];

% Flip angle rotation
R_alpha=[1  0           0          0;
         0  cos(alpha)  sin(alpha) 0;
         0 -sin(alpha)  cos(alpha) 0;
         0  0           0          1];

% Homogeneous expression for RF MATRIX
dB1=R_theta.'*R_alpha*R_theta;

return
```



$$R^{90^\circ}_{90^\circ}$$

$$\mathbf{R}^{90^\circ}_{90^\circ} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Thanks

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