

# Bloch Equations and Relaxation I

M219 - Principles and Applications of MRI

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1/15/2025

# Course Overview

- 2025 course schedule
  - [https://mrrl.ucla.edu/pages/m219\\_2025](https://mrrl.ucla.edu/pages/m219_2025)
- Assignments
  - Homework #1 due on 1/29
- TA office hours, Mondays 4-6pm
- Office hours, Fridays 10-11am
  - In-person (Ueberroth, 1417B) or Zoom

# Last Time...

$$\vec{\tau} = \vec{\mu} \times \vec{B} \quad \vec{S} = \vec{r} \times \vec{p}$$

$$\vec{\tau} = \frac{d\vec{S}}{dt} \quad \vec{\mu} = \gamma \vec{S}$$

$$\frac{d\vec{\mu}}{dt} = \vec{\mu} \times \gamma \vec{B}$$

Equation of Motion for a Magnetic Dipole

$$\vec{M} = \sum_{n=1}^{N_{total}} \vec{\mu}_n$$

$$M_x(t) = M_x^0 \cos(\gamma B_0 t) + M_y^0 \sin(\gamma B_0 t)$$

$$M_y(t) = -M_x^0 \sin(\gamma B_0 t) + M_y^0 \cos(\gamma B_0 t)$$

$$M_z(t) = M_z^0$$

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$

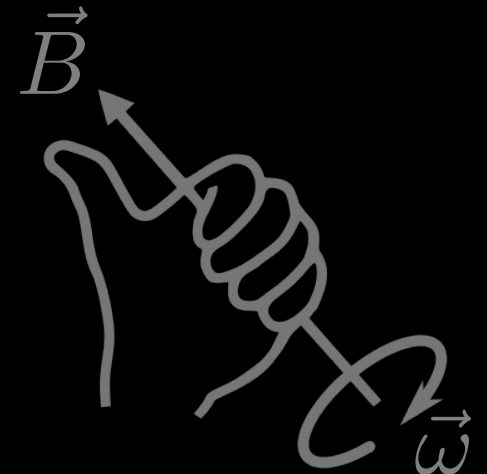
Equation of Motion for the bulk magnetization.

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma (\vec{B}_0)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ M_x & M_y & M_z \\ 0 & 0 & \gamma B_0 \end{vmatrix}$$

$$\vec{B}_0 = B_0 \vec{k}$$

# Free Precession w/o Relaxation

$$\mathbf{R}_z(\omega_0 t) = \begin{bmatrix} \cos \omega_0 t & \sin \omega_0 t & 0 \\ -\sin \omega_0 t & \cos \omega_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
A diagram illustrating left-handed precession. A vector labeled  $\vec{B}$  points upwards and to the left. A helical path winds around the vector, with a circular arrow at the bottom indicating a clockwise direction of rotation when viewed from above.

**Precession is left-handed (clockwise).**

$$\vec{M}(t) = \mathbf{R}_z(\omega_0 t) \vec{M}^0$$

# Basic RF Pulse

$$\vec{B} = \vec{B}_0 + \vec{B}_1(t)$$

$$\vec{B}_1(t) = B_1^e(t) [\cos(\omega_{RF}t + \theta)\hat{i} - \sin(\omega_{RF}t + \theta)\hat{j}]$$

$$B_1^e(t)$$

**pulse envelope function**

$$\omega_{RF}$$

**excitation carrier frequency**

$$\theta$$

**initial phase angle**

$B_1$  is perpendicular to  $B_0$ .

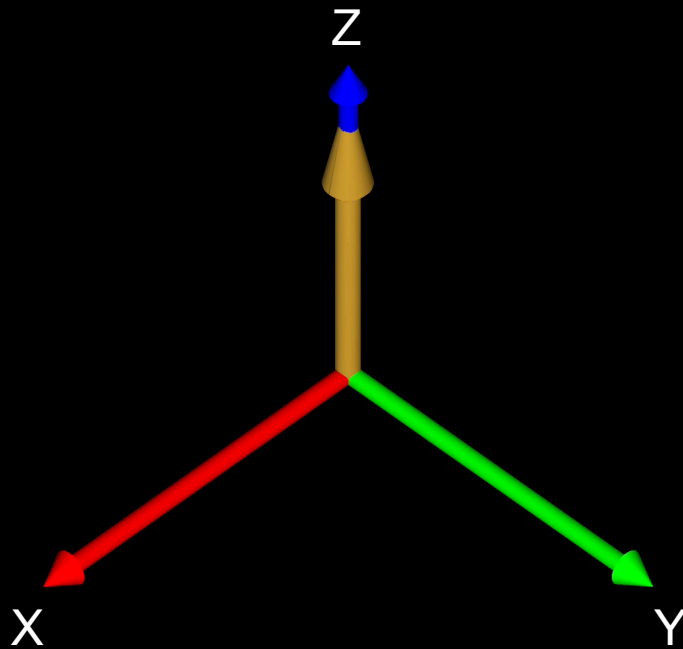
$$\vec{B}_0 = B_0\hat{k}$$

# Rotating Frame

# Lab vs. Rotating Frame

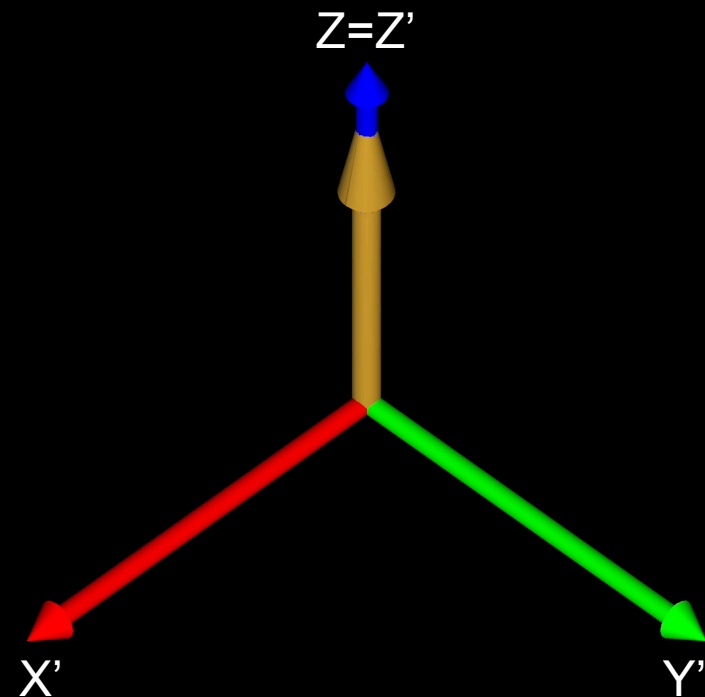
- The rotating frame simplifies the mathematics and permits more intuitive understanding.

90° RF (Laboratory Frame)



*Spins Precess*

90° RF (Rotating Frame)



*Observer Precesses*

**Note:** Both coordinate frames share the same z-axis.

# Combined $B_0$ & $B_1$ Effects

$$\begin{aligned}\frac{d\vec{M}}{dt} &= \vec{M} \times \gamma \vec{B} \\ &= \vec{M} \times \gamma \left( \vec{B}_0 + \vec{B}_1 \right)\end{aligned}$$



# Relationship Between Lab and Rotating Frames

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$

## Rotating Frame Definitions

$$\vec{M}_{rot} \equiv \begin{bmatrix} M_{x'} \\ M_{y'} \\ M_{z'} \end{bmatrix} \quad \vec{B}_{rot} \equiv \begin{bmatrix} B_{x'} \\ B_{y'} \\ B_{z'} \end{bmatrix} \quad \begin{aligned} B_{z'} &\equiv B_z \\ M_{z'} &\equiv M_z \end{aligned}$$

$$\vec{M}_{lab}(t) = R_Z(\omega_{RF}t) \cdot \vec{M}_{rot}(t)$$

Bulk magnetization components in the rotating frame.

$$\vec{B}_{lab}(t) = R_Z(\omega_{RF}t) \cdot \vec{B}_{rot}(t)$$

Applied B-field components in the rotating frame.

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B} \quad \longrightarrow \quad \frac{d\vec{M}_{rot}}{dt} = \vec{M}_{rot} \times \gamma \vec{B}_{eff}$$

# Bloch Equation (Rotating Frame)

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$

Equation of motion for an ensemble of spins (isochromats).  
[Laboratory Frame]

$$\frac{d\vec{M}_{rot}}{dt} = \vec{M}_{rot} \times \gamma \left( \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot} \right)$$

Equation of motion for an ensemble of spins (isochromats).  
[Rotating Frame]

$$\vec{B}_{eff} \equiv \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot} \quad \vec{\omega}_{rot} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{RF} \end{pmatrix}$$

↑  
Effective B-field that  $M$  experiences in the rotating frame.

↑  
Fictitious field that demodulates the apparent effect of  $B_0$ .

↑  
Applied B-field in the rotating frame.

# Bloch Equation (Rotating Frame)

$$\vec{B}(t) = B_0 \hat{k} + B_1^e(t) [\cos(\omega_{RF}t + \theta) \hat{i} - \sin(\omega_{RF}t + \theta) \hat{j}]$$

$$\vec{B}_{lab}(t) = \begin{pmatrix} B_1^e(t) \cos(\omega_{RF}t + \theta) \\ -B_1^e(t) \sin(\omega_{RF}t + \theta) \\ B_0 \end{pmatrix} \quad \vec{B}_{rot}(t) = \begin{pmatrix} B_1^e(t) \cos \theta \\ -B_1^e(t) \sin \theta \\ B_0 \end{pmatrix}$$

$$\vec{B}_{eff} \equiv \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot} \quad \vec{\omega}_{rot} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{RF} \end{pmatrix}$$

Effective B-field that  $M$  experiences in the rotating frame.

Fictitious field that demodulates the apparent effect of  $B_0$ .

Applied B-field in the rotating frame.

# Bloch Equation (Rotating Frame)

$$\vec{B}_{eff} \equiv \frac{\vec{\omega}_{rot}}{\gamma} + \vec{B}_{rot}$$

Assume no RF phase ( $\theta = 0$ )

$$\vec{B}_{rot}(t) = \begin{pmatrix} B_1^e(t) \\ 0 \\ B_0 \end{pmatrix} \quad \vec{\omega}_{rot} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{RF} \end{pmatrix}$$

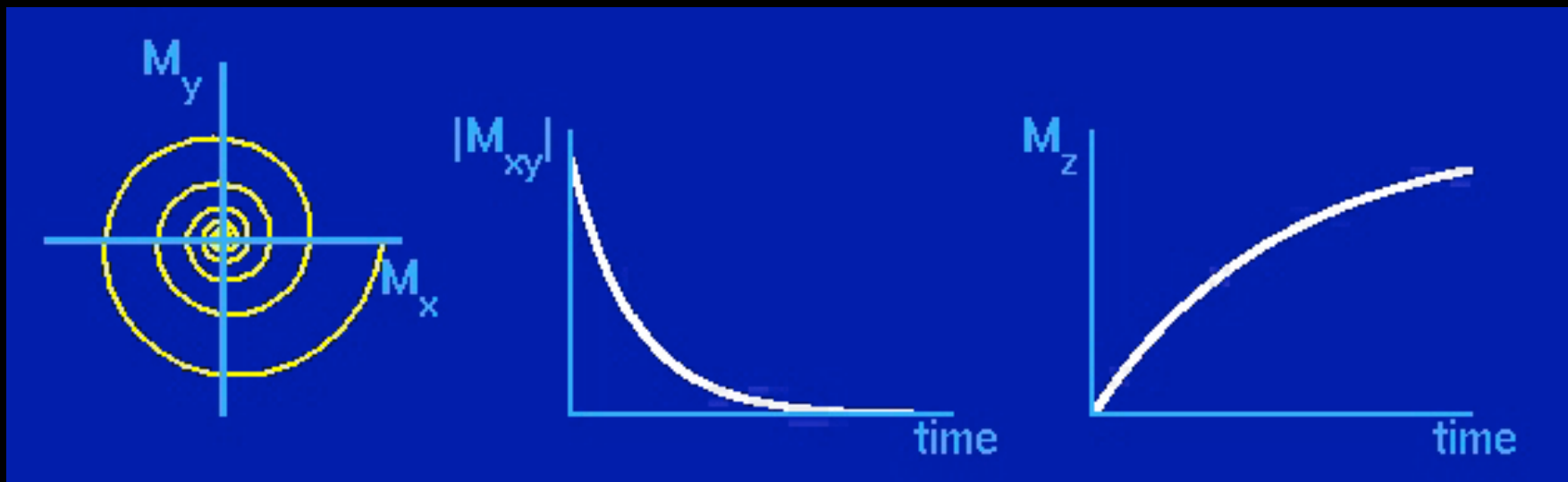
$$\vec{B}_{eff}(t) = \begin{pmatrix} B_1^e(t) \\ 0 \\ B_0 \end{pmatrix} \begin{matrix} \\ \\ \omega_{RF} \\ \gamma \end{matrix}$$

To the Board

# $T_1$ & $T_2$ Relaxation

# Relaxation

- Magnetization returns exponentially to equilibrium:
  - Longitudinal recovery time constant is T1
  - Transverse decay time constant is T2
- Relaxation and precession are independent



# T<sub>1</sub> Relaxation

- Longitudinal or spin-lattice relaxation
  - Typically, (10s ms) < T<sub>1</sub> < (100s ms)
- T<sub>1</sub> is long for
  - Small molecules (water)
  - Large molecules (proteins)
- T<sub>1</sub> is short for
  - Fats and intermediate-sized molecules
- T<sub>1</sub> increases with increasing B<sub>0</sub>
- T<sub>1</sub> decreases with contrast agents

Short T<sub>1</sub>s are bright on T<sub>1</sub>-weighted image

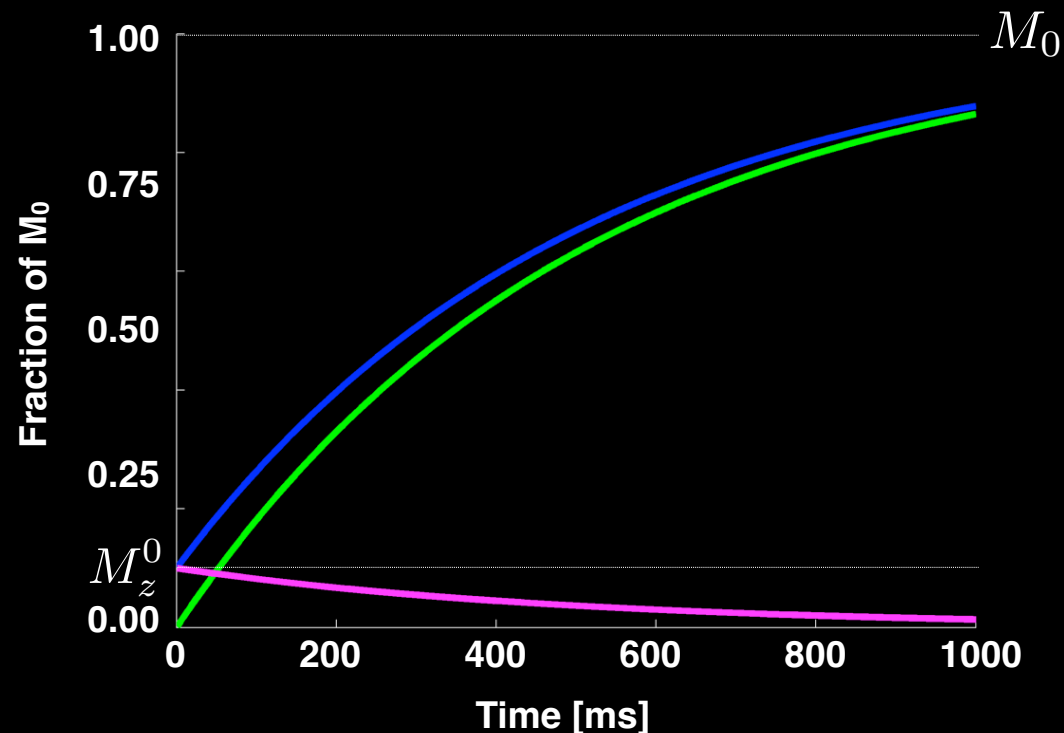


# T<sub>1</sub> Relaxation

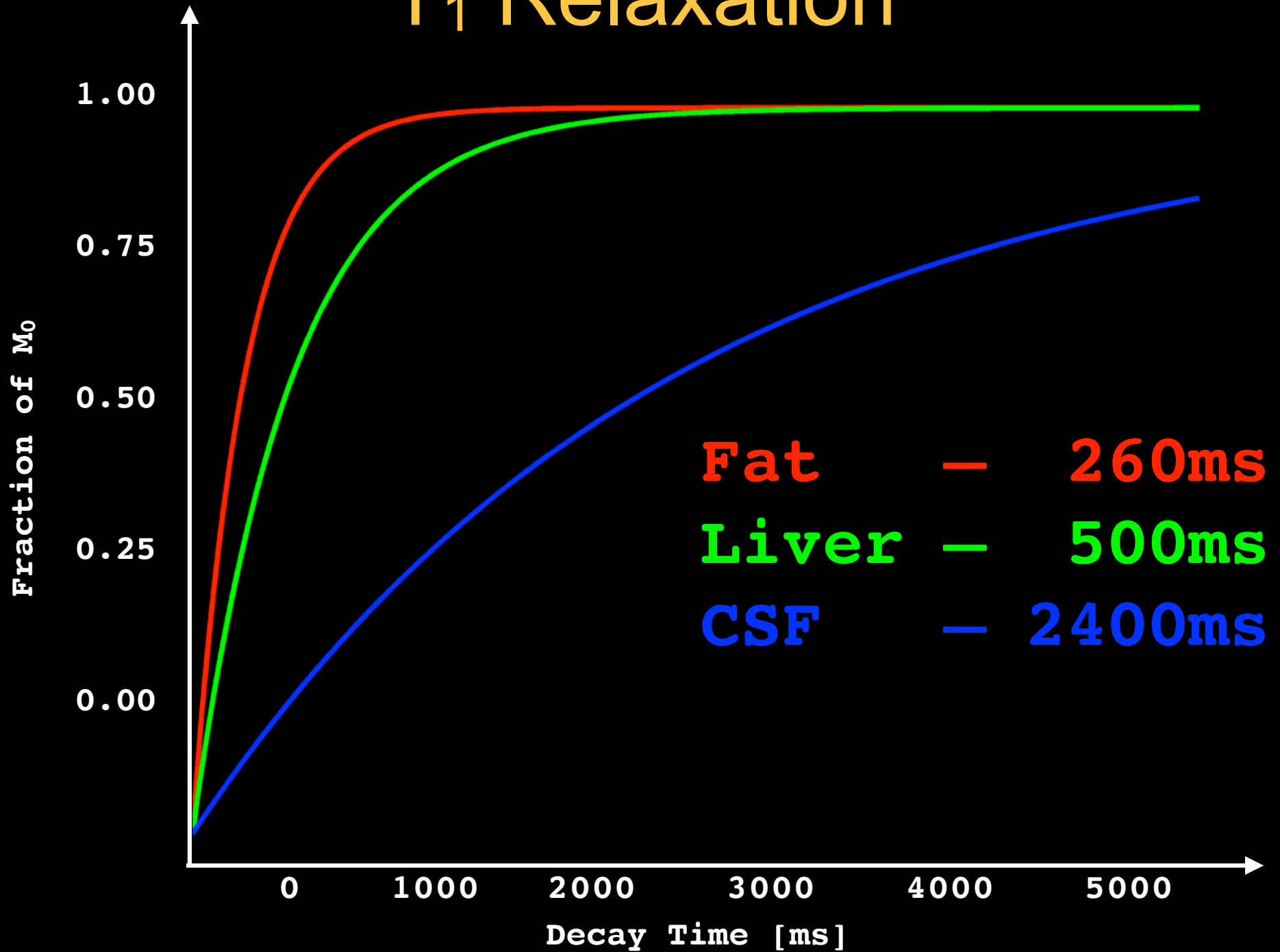
Free Precession in the Lab *or* Rotating Frame with Relaxation

$$M_{z'}(t) = \underbrace{M_z^0}_{\text{Net Magnetization}} e^{-t/T_1} + \underbrace{M_0}_{\text{Prepared Magnetization Decays } (M_z^0)} (1 - e^{-t/T_1})$$

Net Magnetization      Prepared Magnetization Decays ( $M_z^0$ )      Return to Thermal Equilibrium ( $M_0$ )

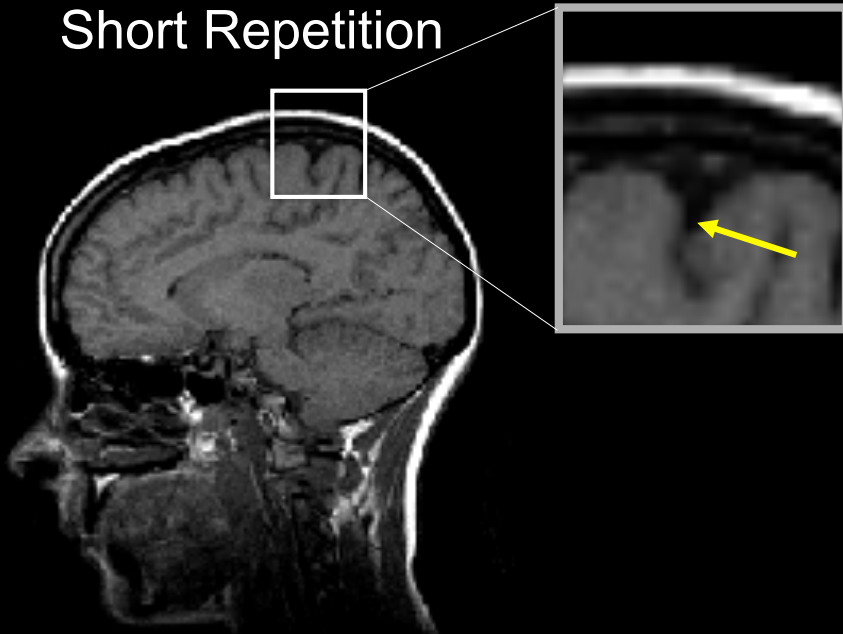


# T<sub>1</sub> Relaxation

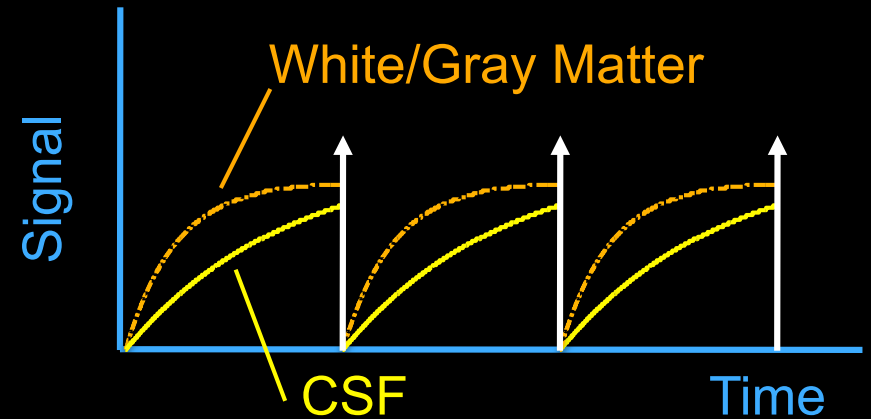
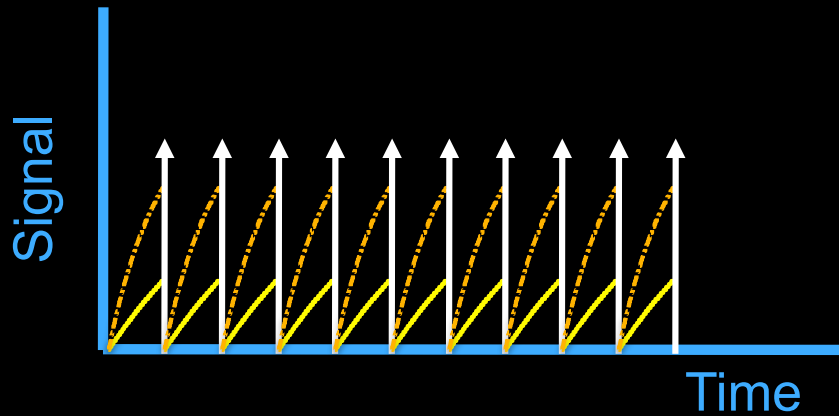
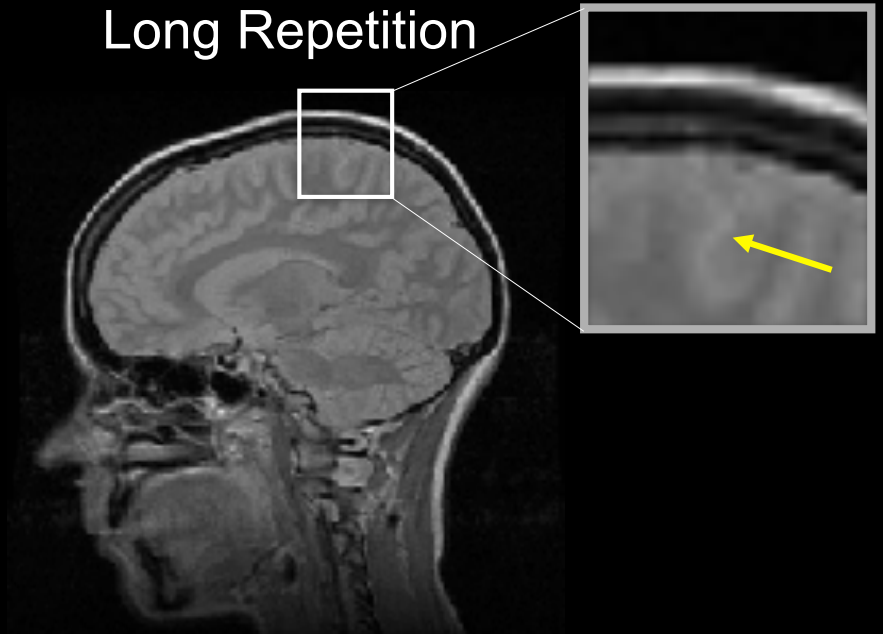


# T<sub>1</sub> Contrast

Short Repetition



Long Repetition



# T<sub>2</sub> Relaxation

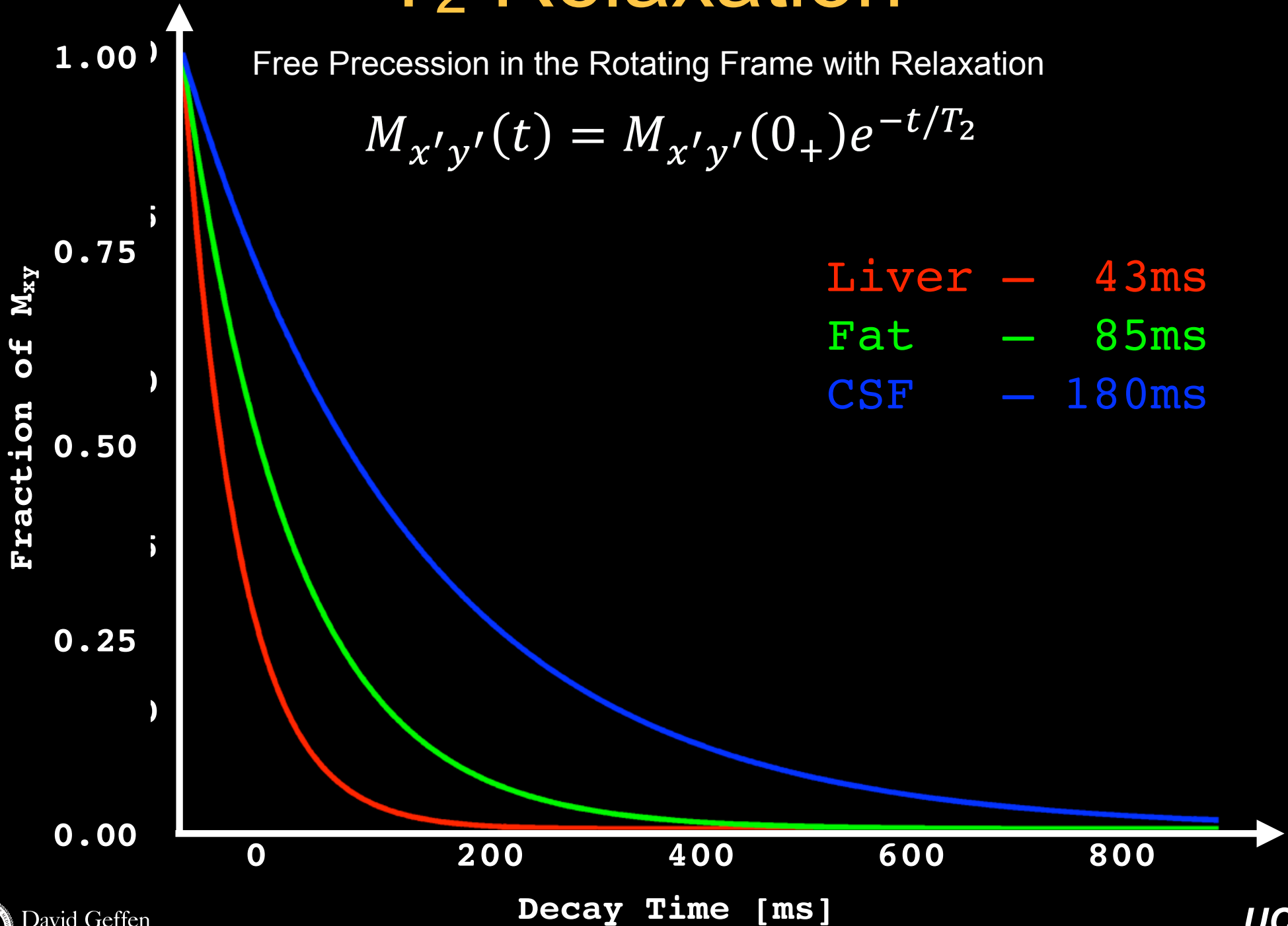
- Transverse or spin-spin relaxation
  - Molecular interaction causes spin dephasing
  - Typically, T<sub>2</sub> < (10s ms)
- Increasing molecular size, decrease T<sub>2</sub>
  - Fat has a short T<sub>2</sub>
- Increasing molecular mobility, increases T<sub>2</sub>
  - Liquids (CSF, edema) have long T<sub>2</sub>s
- Increasing molecular interactions, decreases T<sub>2</sub>
  - Solids have short T<sub>2</sub>s
- T<sub>2</sub> relatively independent of B<sub>0</sub>

Long T<sub>2</sub> is bright on T<sub>2</sub> weighted image

# T<sub>2</sub> Relaxation

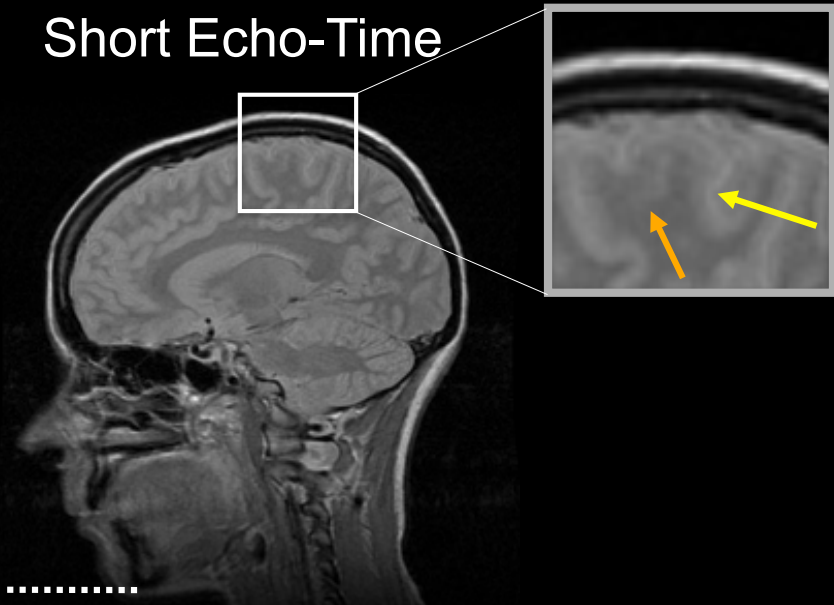
Free Precession in the Rotating Frame with Relaxation

$$M_{x'y'}(t) = M_{x'y'}(0_+)e^{-t/T_2}$$

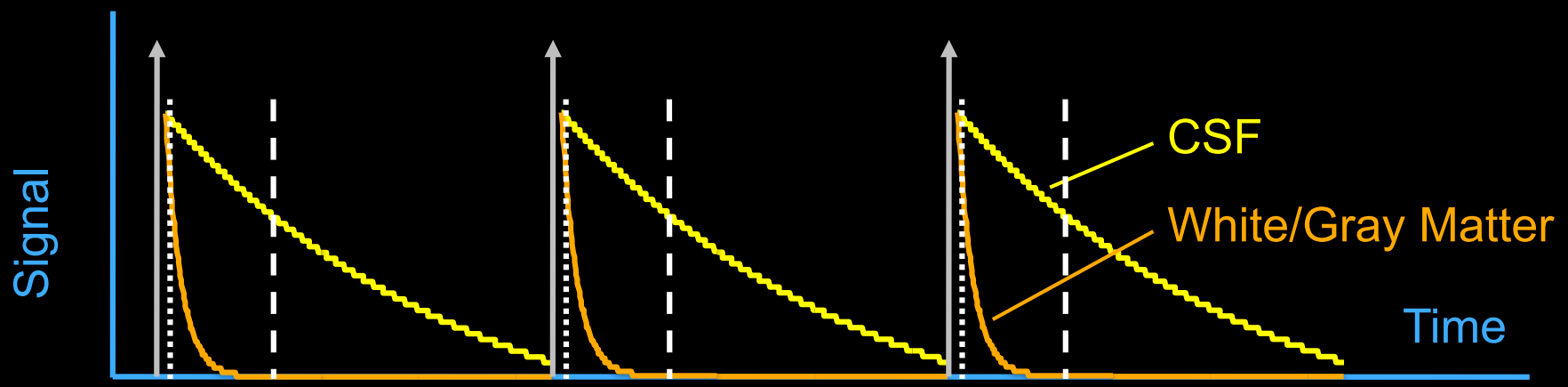
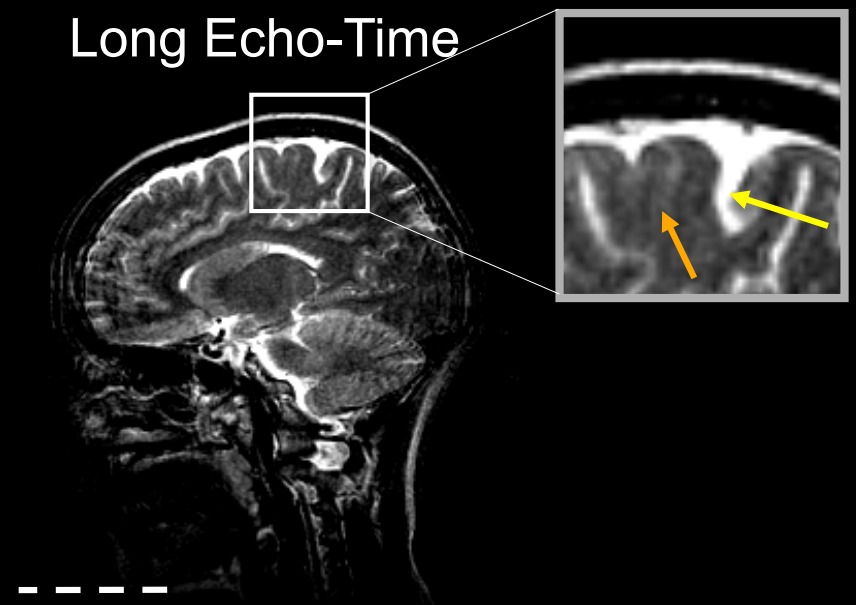


# T2 Contrast

Short Echo-Time



Long Echo-Time



# T<sub>1</sub> and T<sub>2</sub> Values @ 1.5T

Tissue	T <sub>1</sub> [ms]	T <sub>2</sub> [ms]
gray matter	925	100
white matter	790	92
muscle	875	47
fat	260	85
kidney	650	58
liver	500	43
CSF	2400	180

Each tissue has “unique” relaxation properties, which enables “soft tissue contrast”.

# $T_2^*$ Relaxation



# $T_2^*$ Relaxation

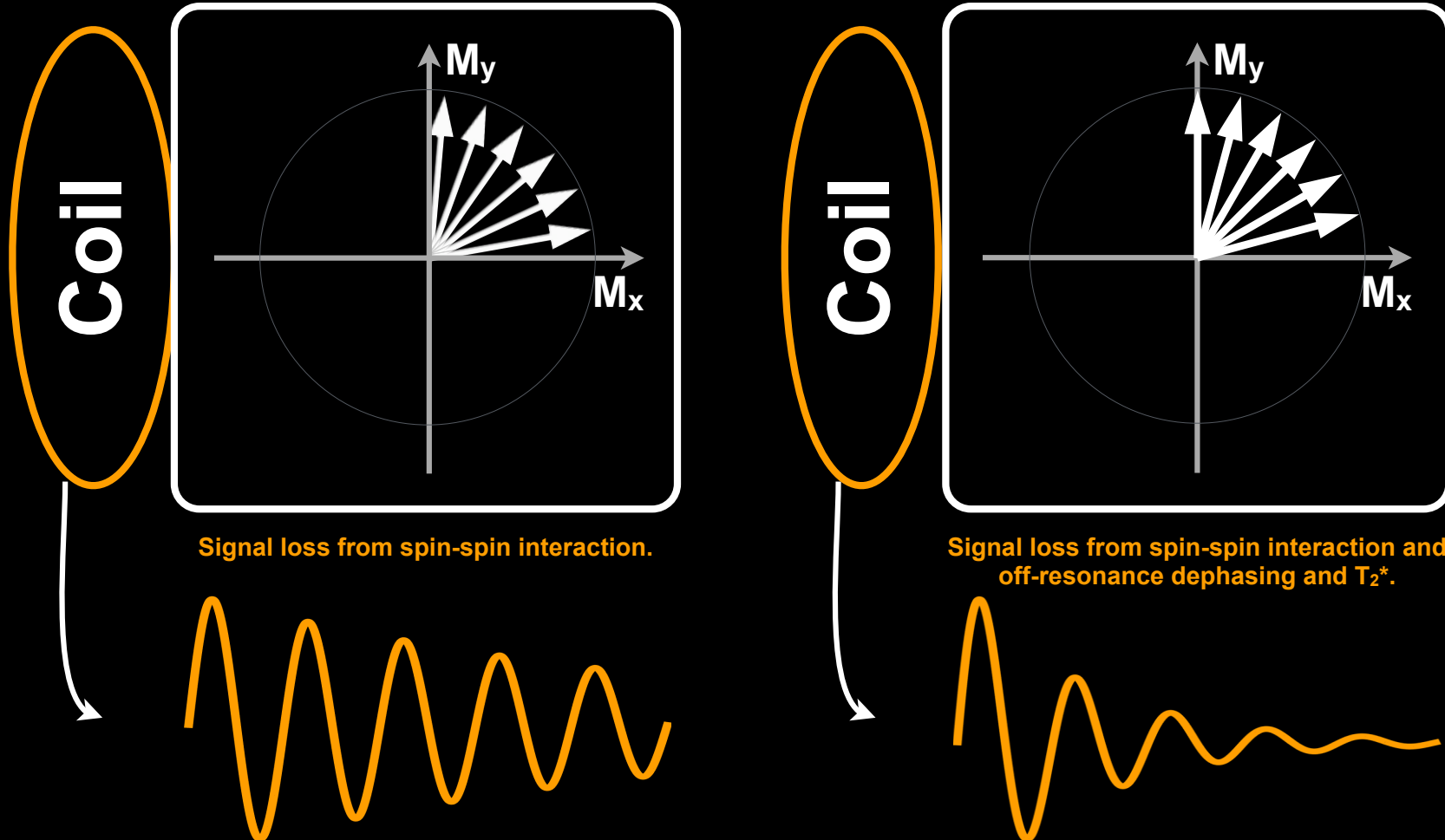
$$\frac{1}{T_2^*} = \frac{1}{T_2} + \gamma \Delta B_0$$

- $T_2^*$  is “observed” transverse relaxation time constant
- $T_2^*$  consists of irreversible spin-spin ( $T_2$ ) dephasing and reversible intravoxel spin dephasing due to off-resonance
- Sources of off-resonance:
  - $B_0$  inhomogeneity
  - susceptibility differences (e.g. air spaces)

# $T_2$ versus $T_2^*$

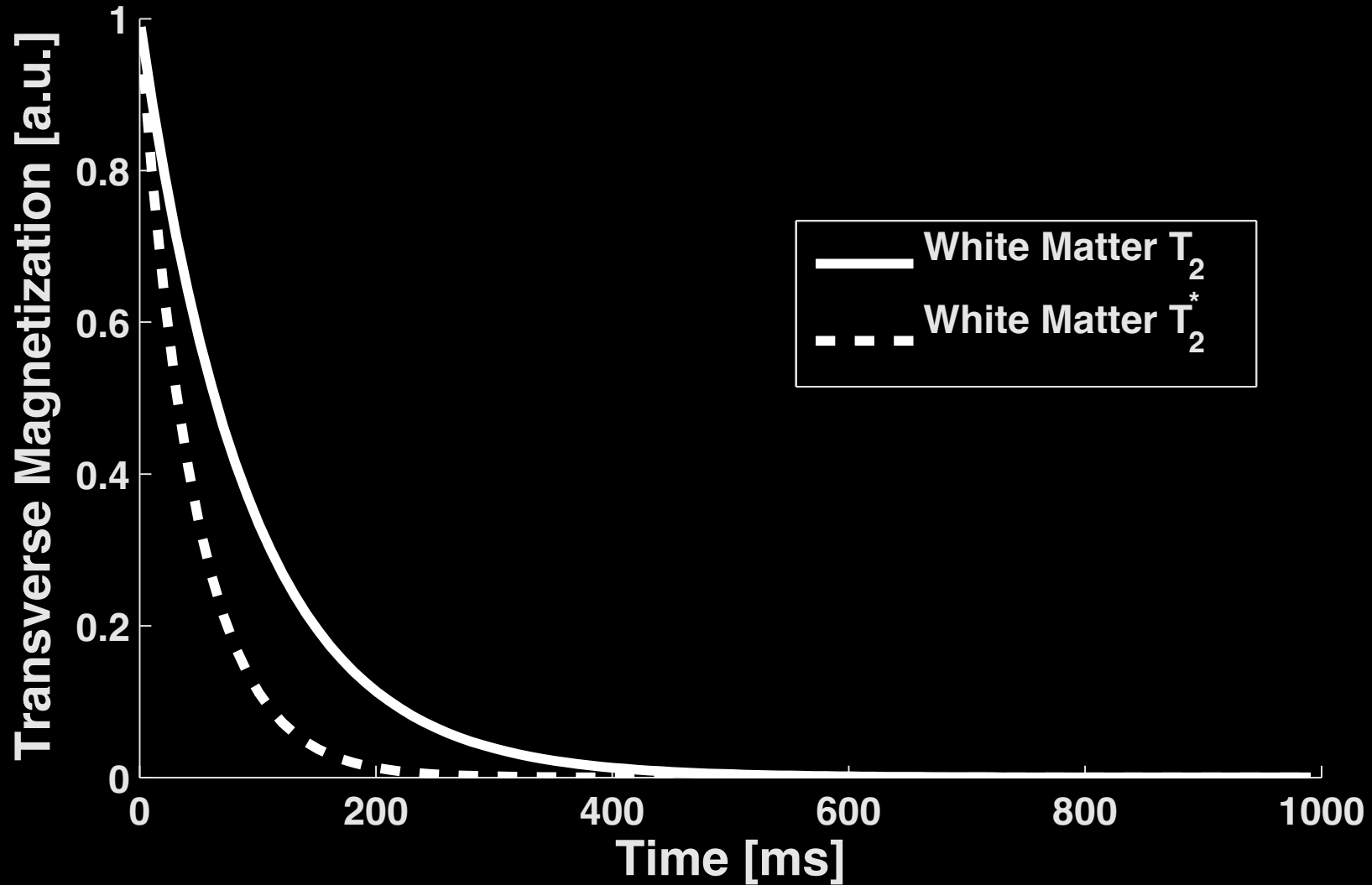
$T_2$  Decay

$T_2^*$  Decay



$T_2^*$  is signal loss from spin dephasing and  $T_2$

$T_2^* < T_2$  (always!)



# Bloch Equations with Relaxation

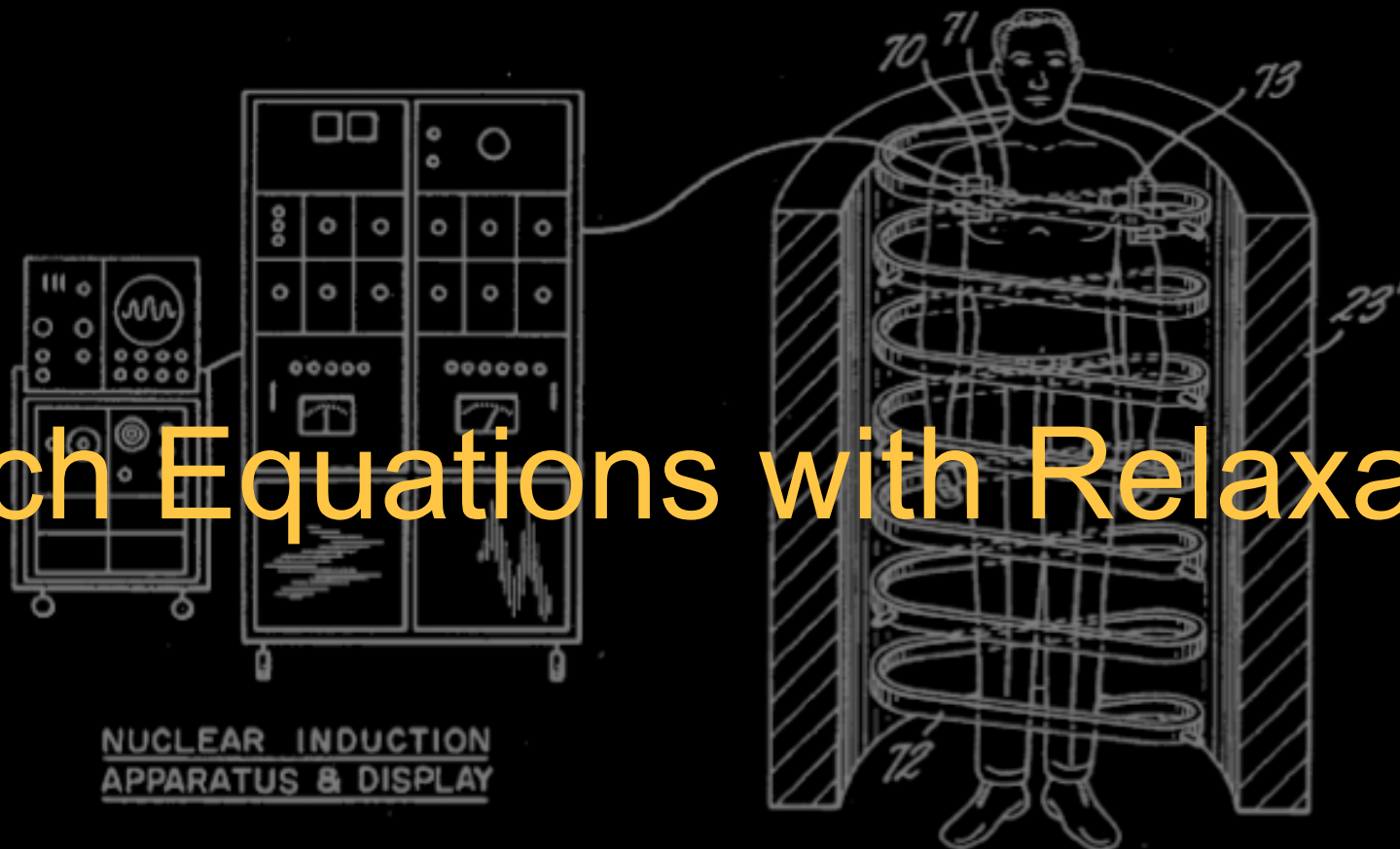


FIG. 2



# Bloch Equations with Relaxation

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_0) \hat{k}}{T_1}$$

- **Differential Equation**
  - Ordinary, Coupled, Non-linear
- **No analytic solution, in general.**
  - Analytic solutions for simple cases.
  - Numerical solutions for all cases.
- **Phenomenological**
  - Exponential behavior is an approximation.

# Bloch Equations - Lab Frame

$$\frac{d\vec{M}}{dt} = \underbrace{\vec{M} \times \gamma \vec{B}}_{\text{Precession}} - \underbrace{\frac{M_x \hat{i} + M_y \hat{j}}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_z - M_0) \hat{k}}{T_1}}_{\text{Longitudinal Relaxation}}$$

- Precession
  - Magnitude of M unchanged
  - Phase (rotation) of M changes due to B
- Relaxation
  - $T_1$  changes are slow O(100ms)
  - $T_2$  changes are fast O(10ms)
  - Magnitude of M can be ZERO
- Diffusion
  - Spins are thermodynamically driven to exchange positions.
  - Bloch-Torrey Equations

# Bloch Equations – Rotating Frame

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \underbrace{\gamma \vec{M}_{rot} \times \vec{B}_{eff}}_{\text{“Precession”}} - \underbrace{\frac{M_{x'} \vec{i}' + M_{y'} \vec{j}'}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_{z'} - M_0) \vec{k}'}{T_1}}_{\text{Longitudinal Relaxation}}$$

$$\vec{B}_{eff} \triangleq \frac{\vec{\omega}}{\gamma} + \vec{B}_{rot}$$

↑  
Effective B-field that M experiences in the rotating frame

↑  
The applied B<sub>0</sub> and B<sub>1</sub> field in the rotating frame

↑  
Fictitious field created by the rotating frame that demodulates the apparent effect of B<sub>0</sub>

# Free Precession in the Rotating Frame with Relaxation



# Free Precession in the Rotating Frame

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \gamma \vec{M}_{rot} \times \vec{B}_{eff} - \frac{M_{x'} \vec{i}' + M_{y'} \vec{j}'}{T_2} - \frac{(M_{z'} - M_0) \vec{k}'}{T_1}$$

$$\vec{B}_{eff} \triangleq \frac{\vec{\omega}}{\gamma} + \vec{B}_{rot}$$

$$\vec{\omega}_{rot} = \vec{\omega} = -\gamma B_0 \hat{k} \quad \vec{B}_{rot} = B_0 \hat{k}$$

$$\vec{B}_{eff} = \vec{0}$$

$$\frac{\partial \vec{M}_{rot}}{\partial t} = - \frac{M_{x'} \vec{i}' + M_{y'} \vec{j}'}{T_2} - \frac{(M_{z'} - M_0) \vec{k}'}{T_1}$$

# Free Precession in the Rotating Frame

$$\frac{\partial \vec{M}_{rot}}{\partial t} = - \underbrace{\frac{M_{x'} \vec{i}' + M_{y'} \vec{j}'}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_{z'} - M_0) \vec{k}'}{T_1}}_{\text{Longitudinal Relaxation}}$$

- **No precession**
- **T<sub>1</sub> and T<sub>2</sub> Relaxation**
- **Drop the diffusion term**
- **System of first order, linear, separable ODEs!**

# Free Precession in the Rotating Frame

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \underbrace{-\frac{M_{x'}\vec{i}' + M_{y'}\vec{j}'}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_{z'} - M_0)\vec{k}'}{T_1}}_{\text{Longitudinal Relaxation}}$$

**Solution:**

$$M_{z'}(t) = M_z^0 e^{-t/T_1} + M_0(1 - e^{-t/T_1})$$

$$M_{x'y'}(t) = M_{x'y'}(0_+) e^{-t/T_2}$$

# Forced Precession in the Rotating Frame with Relaxation

# Forced Precession in the Rot. Frame with Relaxation

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \gamma \vec{M}_{rot} \times \vec{B}_{eff} - \frac{M_{x'} \vec{i}' + M_{y'} \vec{j}'}{T_2} - \frac{(M_{z'} - M_0) \vec{k}'}{T_1}$$

$$\vec{B}_{eff} \triangleq \frac{\vec{\omega}}{\gamma} + \vec{B}_{rot}$$

$$\vec{\omega}_{rot} = \vec{\omega} = -\gamma B_0 \hat{k} \quad \vec{B}_{rot} = B_0 \hat{k} + B_1^e(t) \hat{i}'$$

$$\vec{B}_{eff} = B_1^e(t) \hat{i}'$$

# Forced Precession in the Rot. Frame with Relaxation

$$\frac{\partial \vec{M}_{rot}}{\partial t} = \gamma \vec{M}_{rot} \times \vec{B}_{eff} - \frac{M_{x'} \vec{i}' + M_{y'} \vec{j}'}{T_2} - \frac{(M_{z'} - M_0) \vec{k}'}{T_1}$$

$$\vec{B}_{eff} = B_1^e(t) \hat{i}'$$

- **B1 induced nutation**
- **T<sub>1</sub> and T<sub>2</sub> Relaxation**
- **Drop the diffusion term**
- **System or first order, linear, coupled PDEs!**
- **When does this equation apply?**

# Forced Precession in the Rotating Frame with Relaxation

- RF pulses are short
  - $100\mu\text{s}$  to  $5\text{ms}$
- Relaxation time constants are long
  - $T_1$   $O(100\text{s})$  ms
  - $T_2$   $O(10\text{s})$  ms
- Complicated Coupling
- Best suited for simulation

# Free? Forced? Relaxation?

- **We've considered all combinations of:**
  - Free and forced precession
  - With and without relaxation
  - Laboratory and rotating frames
- **Which one's concern M219 the most?**
  - Free precession in the rotating frame with relaxation
  - Forced precession in the rotating frame without relaxation.
- **We can, in fact, simulate all of them...**



# Questions?

- Related reading materials
  - Nishimura - Chap 4 and 5
  - Nishimura - Chap 6 (Appendix I)

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